Constraint Semantics and Constraint Solving

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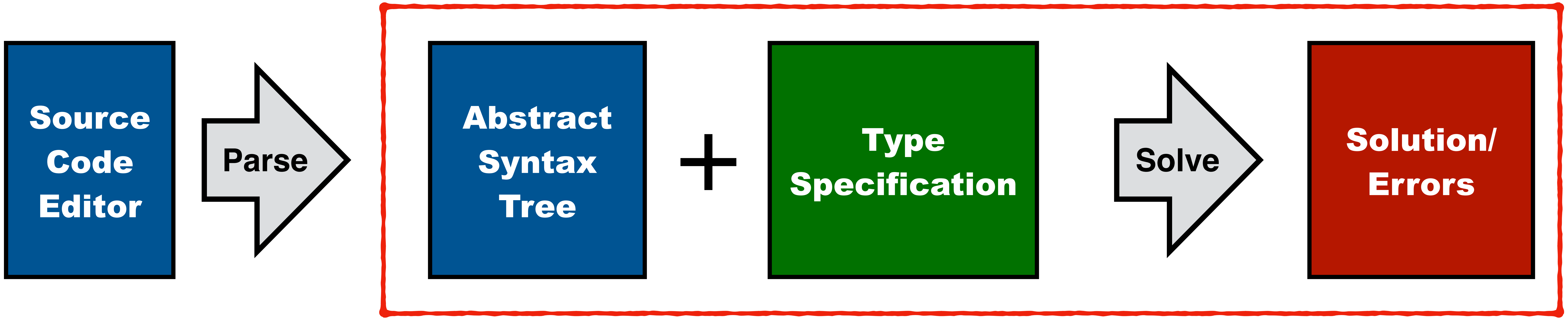
Eelco Visser

TU Delft

CS4200 | Compiler Construction | October 1, 2020

This lecture



- Type checking with type specifications
- Semantics of a type specification
- Type checking algorithms
- Constraint solving for type specifications
- Term equality and unification

Reading Material

The following papers add background, conceptual exposition, and examples to the material from the slides. Some notation and technical details have been changed; check the documentation.

This paper describes the next generation of the approach.

Addresses (previously) open issues in expressiveness of scope graphs for type systems:

- Structural types
- Generic types

Addresses open issue with staging of information in type systems.

Introduces Statix DSL for definition of type systems.

OOPSLA 2018

https://doi.org/10.1145/3276484

Scopes as Types

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EELCO VISSER, Delft University of Technology, Netherlands

Scope graphs are a promising generic framework to model the binding structures of programming languages, bridging formalization and implementation, supporting the definition of type checkers and the automation of type safety proofs. However, previous work on scope graphs has been limited to simple, nominal type systems. In this paper, we show that viewing *scopes as types* enables us to model the internal structure of types in a range of non-simple type systems (including structural records and generic classes) using the generic representation of scopes. Further, we show that relations between such types can be expressed in terms of generalized scope graph queries. We extend scope graphs with scoped relations and queries. We introduce Statix, a new domain-specific meta-language for the specification of static semantics, based on scope graphs and constraints. We evaluate the scopes as types approach and the Statix design in case studies of the simply-typed lambda calculus with records, System F, and Featherweight Generic Java.

CCS Concepts: • **Software and its engineering → Semantics**; **Domain specific languages**;

Additional Key Words and Phrases: static semantics, type system, type checker, name resolution, scope graphs, domain-specific language

ACM Reference Format:
Hendrik van Antwerpen, Casper Bach Poulsen, Arjen Rouvoet, and Eelco Visser. 2018. Scopes as Types. *Proc. ACM Program. Lang.* 2, OOPSLA, Article 114 (November 2018), 30 pages. https://doi.org/10.1145/3276484

1 INTRODUCTION

The goal of our work is to support high-level specification of type systems that can be used for multiple purposes, including reasoning (about type safety among other things) and the implementation of type checkers [Visser et al. 2014]. Traditional approaches to type system specification do not reflect the commonality underlying the name binding mechanisms for different languages. Furthermore, operationalizing name binding in a type checker requires carefully staging the traversals of the abstract syntax tree in order to collect information before it is needed. In this paper, we introduce an approach to the declarative specification of type systems that is close in abstraction to traditional type system specifications, but can be directly interpreted as type checking rules. The approach is based on scope graphs for name resolution, and constraints to separate traversal order from solving order.

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2475-1421/2018/11-ART114
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Good introduction to unification, which is the basis of many type inference approaches, constraint languages, and logic programming languages. Read sections 1, and 2.

CHAPTER 8

Unification theory

Franz Baader

Wayne Snyder

SECOND READERS: Paliath Narendran, Manfred Schmidt-Schauss, and Klaus Schulz.

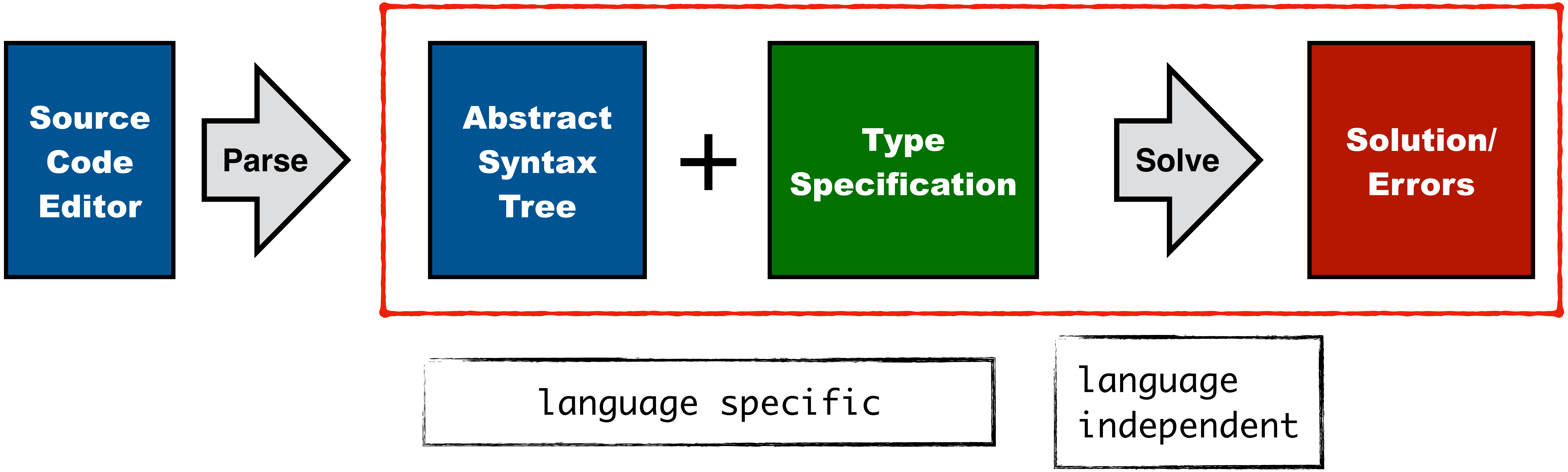
Contents

HANDBOOK OF AUTOMATED REASONING
Edited by Alan Robinson and Andrei Voronkov

Baader et al. “Chapter 8 - Unification Theory.” In Handbook of Automated Reasoning, 445–533. Amsterdam: North-Holland, 2001.

https://www.cs.bu.edu/~snyder/publications/UnifChapter.pdf

Type Checking with Specifications


Source Code Editor
Parse
Abstract Syntax Tree
+
Type Specification
Solve
Solution/ Errors
language specific
language independent

Typing Rules

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Solving

- Given an initial predicate that must hold, ...
- find an assignment for all logical variables, such that the predicate is satisfied

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Approach: reusable solver for the specification language

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- Automatically determine correct resolution order

Constraint Semantics

What gives constraints meaning?

What is the meaning of constraints?

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ty == FUN(ty1,ty2)
Var{x} in s |-> d
ty1 == INT()
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  - Scope graph G
- Describes for every type of constraint when it is satisfied

Semantics of (a Subset of) Statix Constraints

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C = t == t                // equality
  | r in s |-> d          // name resolution (short for query var … in s |-> [d])
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G,ϕ ⊨ C1 /\ C2            if G,ϕ ⊨ C1 and G,ϕ ⊨ C2
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Using the Semantics

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end
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Program constraints

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"i" in #s1 |-> d1
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Unifier ϕ (model)

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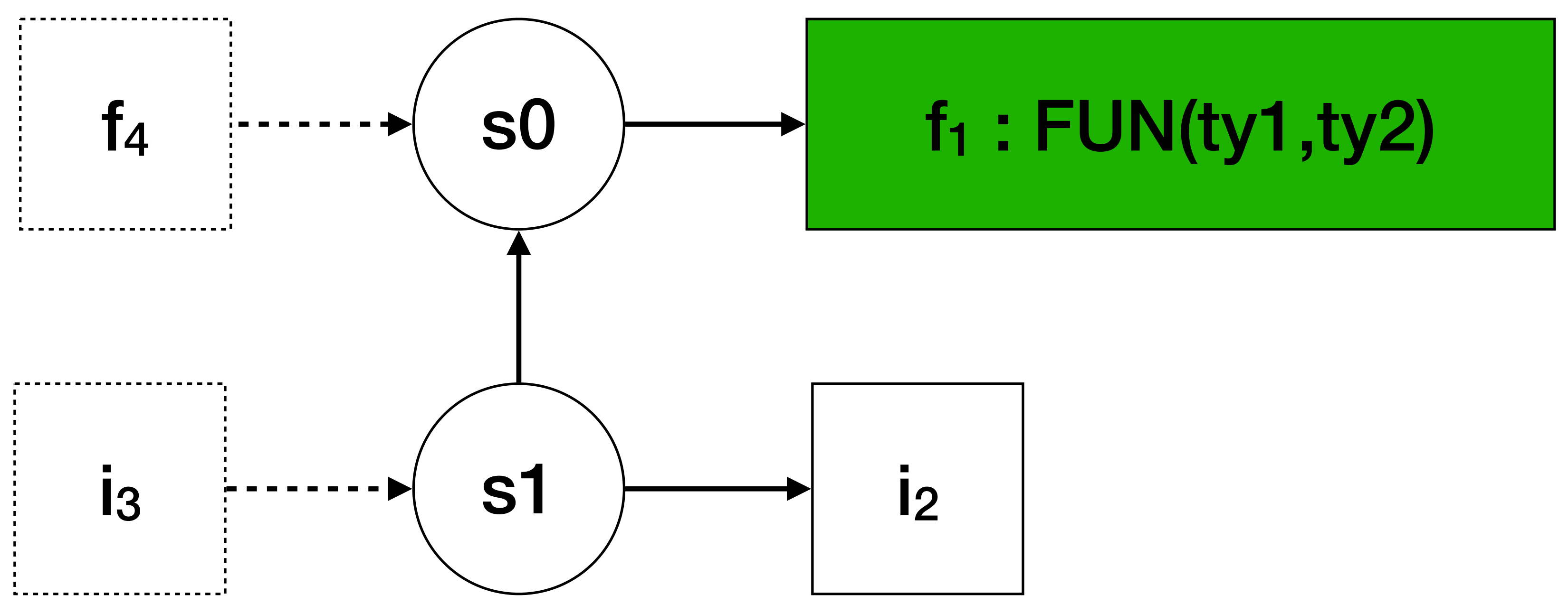

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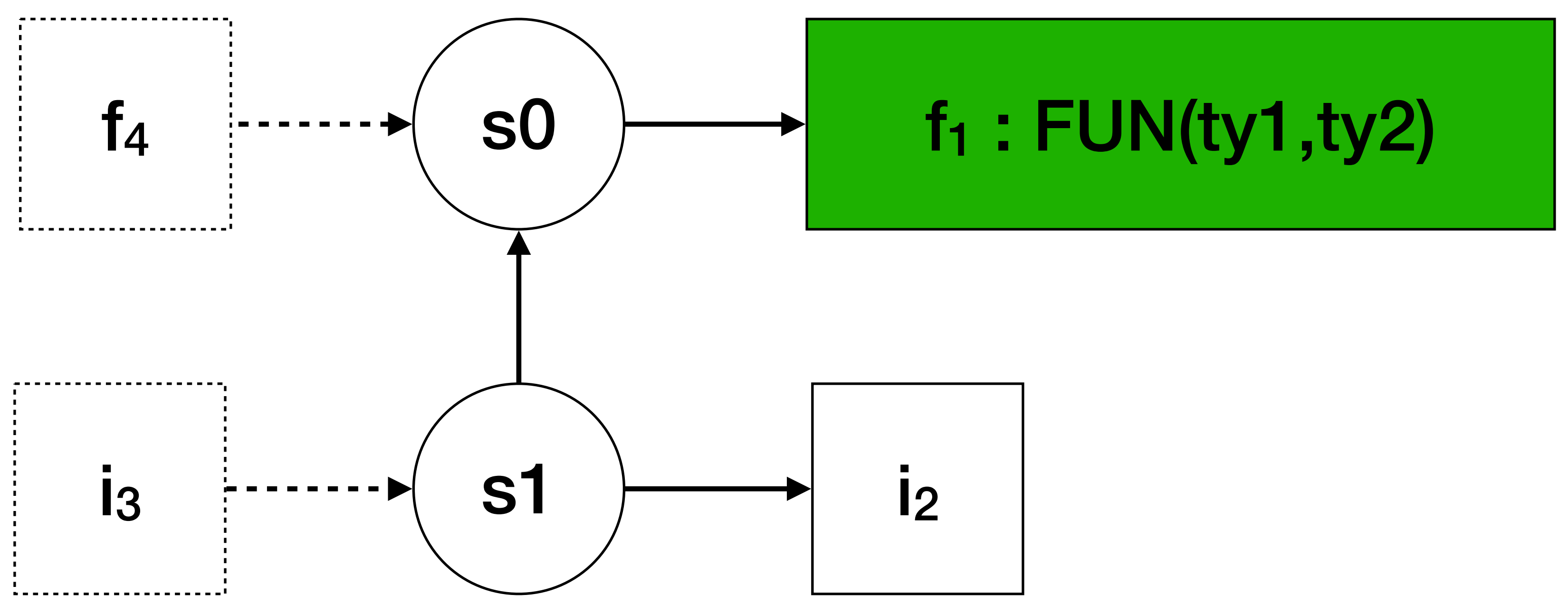

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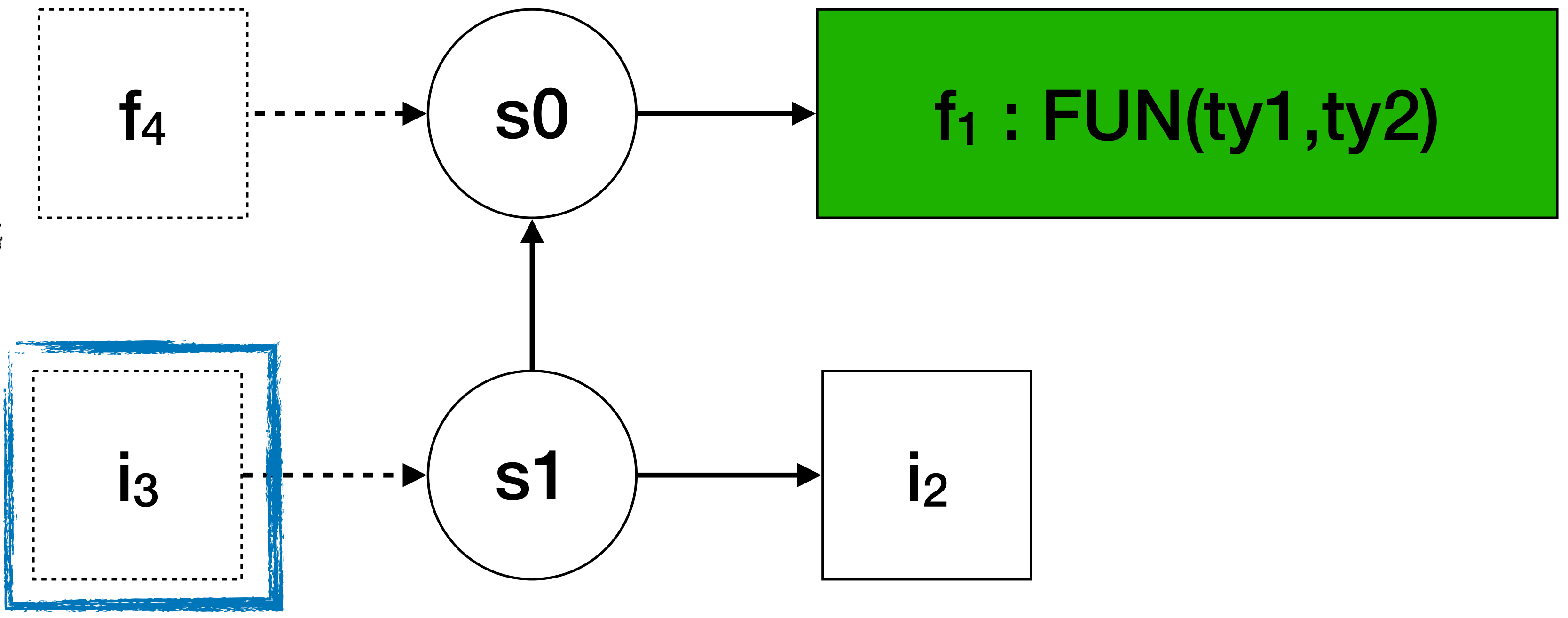
Object language variables



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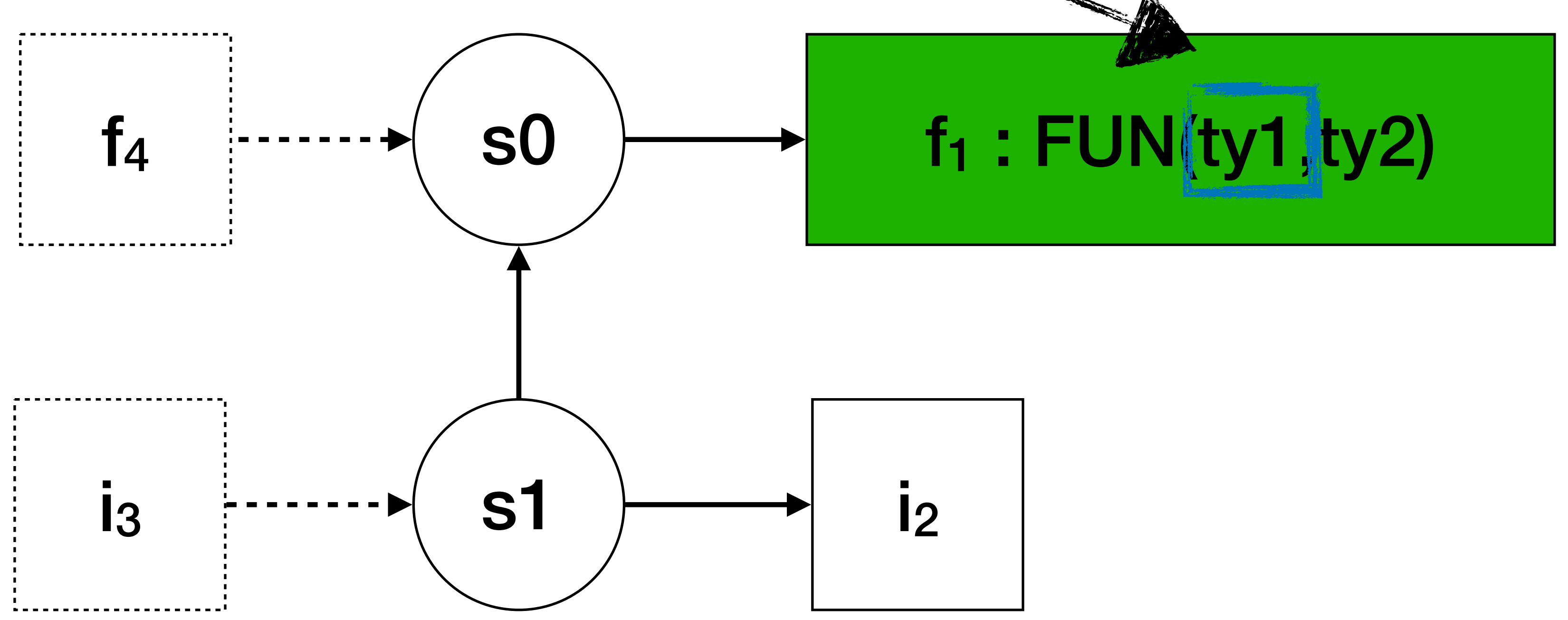
Constraint / logic variables



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Semantics meta-variables

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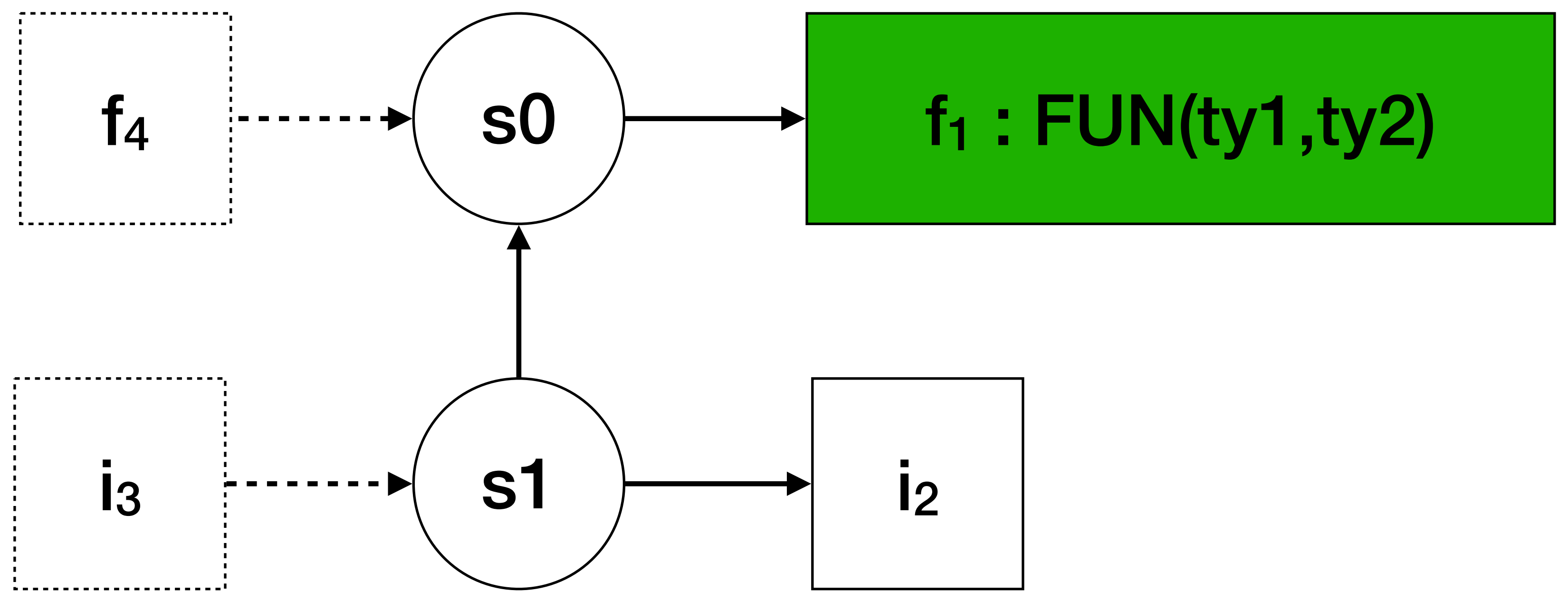

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- IDE (error reporting, code completion, refactoring, …)

How to check types?

What should a type checker do?

- Check that a program is well-typed!
- Resolve names, and check or compute types
- Report useful error messages
- Provide a representation of name and type information
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How are type checkers implemented?

Computing Type of Expression (recap)

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function (a : int) = a + 1
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```
Fun("a", INT(),
    Plus(Var("a"), Int(1)))
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FUN(INT(), INT())
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- Can be executed top down, in premise order
- Could be written almost like this in a functional language

Inferring the Type of a Parameter

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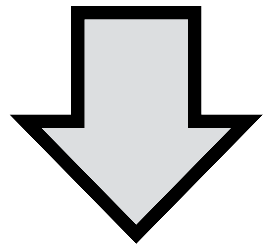

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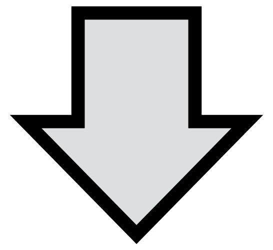

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Checking classes

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        return new C();
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Question

- Does this still work if we introduce nested classes?

Variables and Constraints

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Variables and Constraints

```
function (a [redacted]) = a + 1
```

```
Fun("a", [redacted]
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```

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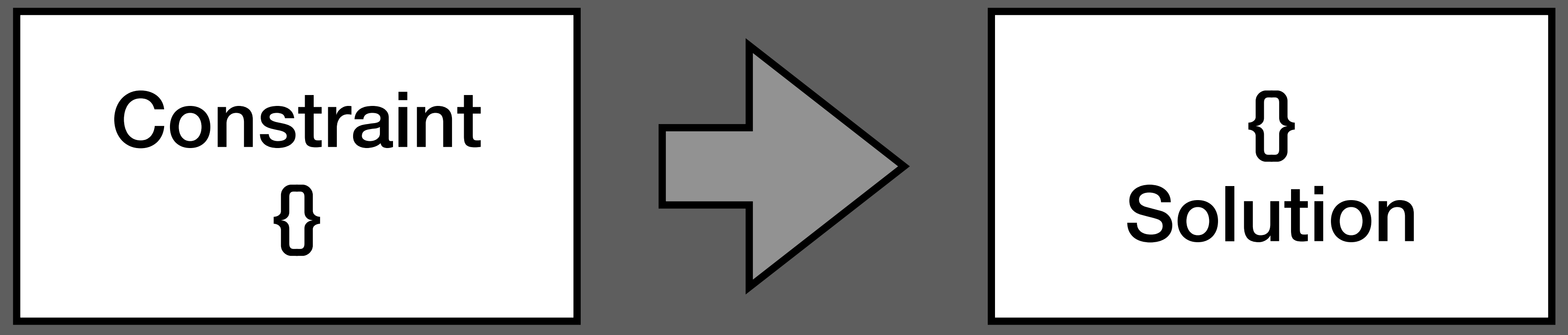
What are the consequences of these challenges?

- The order of computation needs to be more flexible than the AST traversal
- Support explicit logical variables during solving

Solving Constraints

Solving by Rewriting



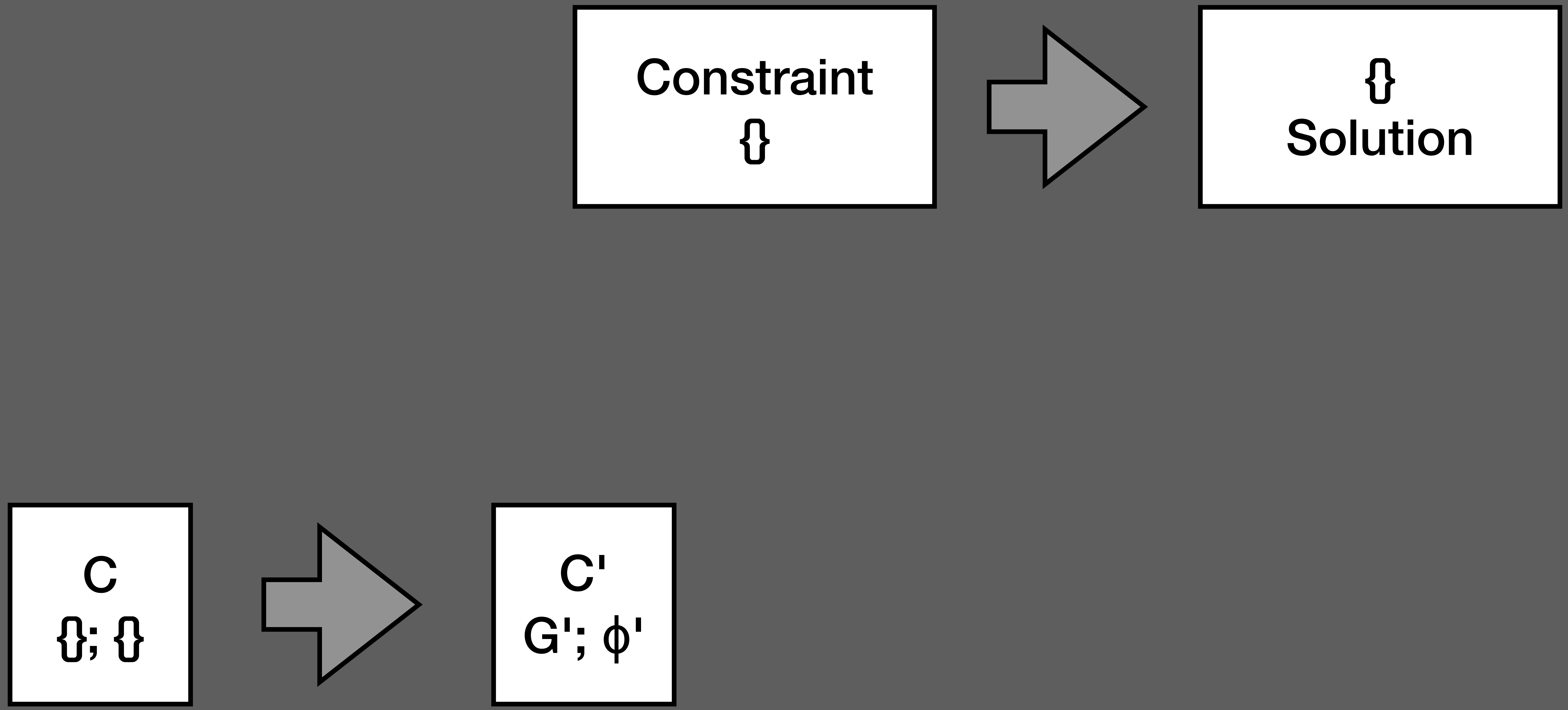
Solving by Rewriting

Constraint
{}

{}
Solution

C
{}; {}

Solving by Rewriting



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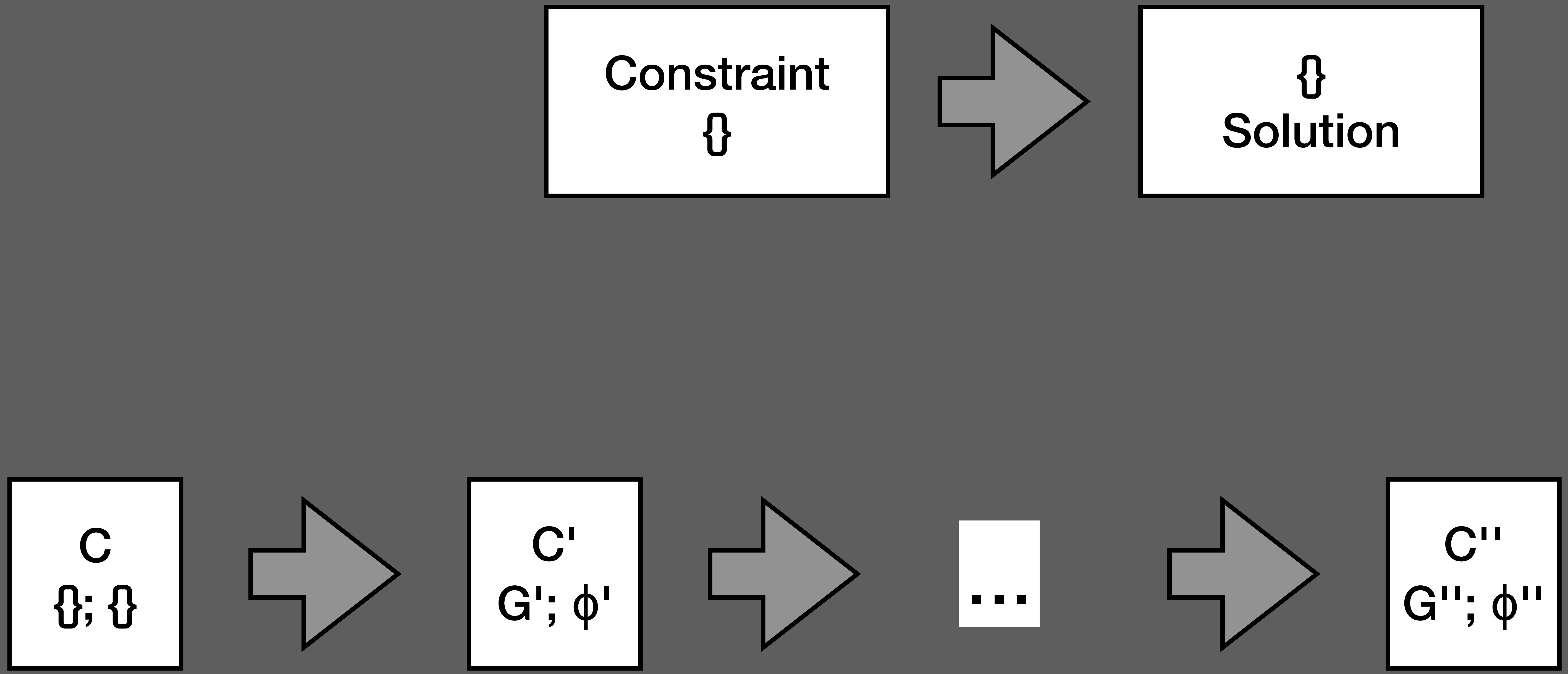

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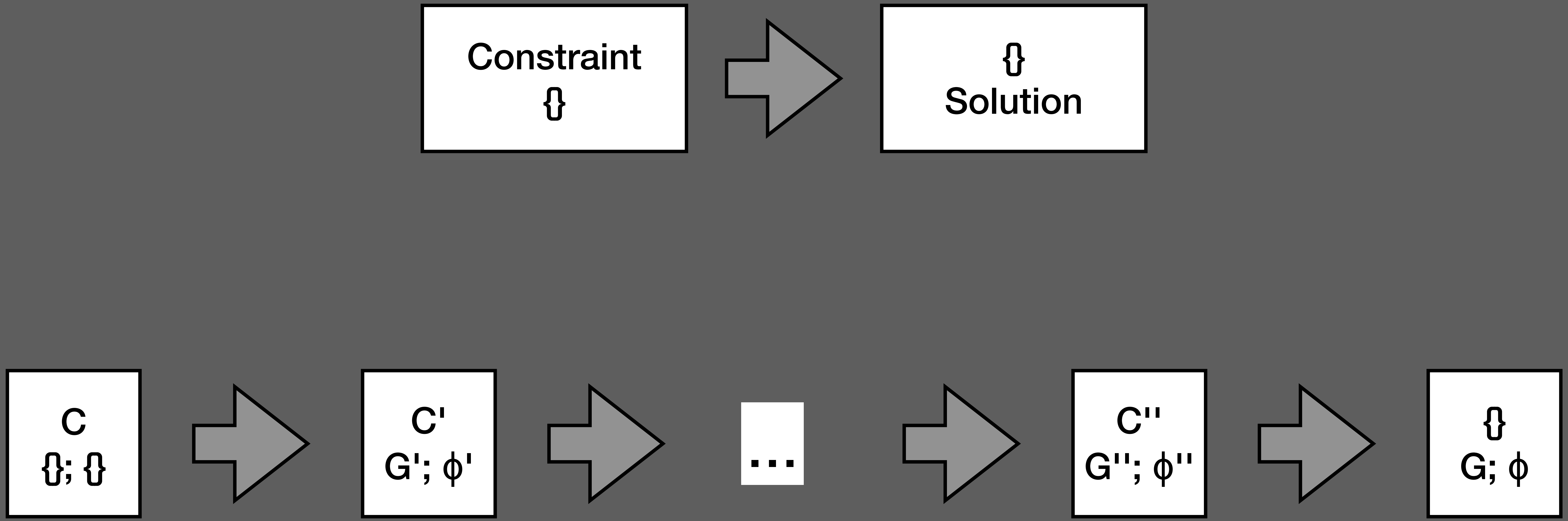

Solving by Rewriting

$$\langle C;\ G,\ \phi\rangle \longrightarrow \langle C;\ G,\ \phi\rangle$$

Solving by Rewriting

```
<C; G, ϕ> ⟶ <C; G, ϕ>
```

```
<t == u, C; G, ϕ> ⟶ <C; G, ϕ'> where unify(ϕ,t,u) = ϕ'
```

Solving by Rewriting

Non-deterministic constraint selection

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<r in s ⟼ t, C; G, ϕ> ⟶ <t ═ d; G, ϕ> where ϕ(r) = Ns{x}, ϕ(s) = #i,
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Scope graph and name resolution algorithm don't have to know about logical variables

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```
def solve(C):
  if <C; {}, {}> ⟶* <{}; G, ϕ>:
    return <G, ϕ>
  else:
    fail
```

Solving by Rewriting

Solver = rewrite system

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- The solution is final if all constraints are eliminated

Does the order matter for the outcome?

- Confluence: the output is the same for any solving order
- Partly true for Statix
  - Up to variable and scope names

Solving by Rewriting

Solver = rewrite system

- Rewrite a constraints set + solution
- Simplifying and eliminating constraints
  - Constraint selecting is non-deterministic
  - Resolution order is controlled by side conditions on rewrite rules
- Rely on (other) solvers and algorithms for base cases
  - Unification for term equality
  - Scope graph resolution
- The solution is final if all constraints are eliminated

Does the order matter for the outcome?

- Confluence: the output is the same for any solving order
- Partly true for Statix
  - Up to variable and scope names
  - Only if all constraints are reduced

Semantics vs Algorithm

What is the difference?

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)
- Semantics describes when a constraint is satisfied by a model

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How are these related?

- Soundness

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)
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How are these related?

- Soundness
  - If the solver returns <G, ϕ>, then $G,\phi \vDash C$

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)
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How are these related?

- Soundness
  - If the solver returns <G, ϕ>, then $G,\phi \vDash C$
- Completeness:

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)
- Semantics describes when a constraint is satisfied by a model

How are these related?

- Soundness
    - If the solver returns <G, ϕ>, then $G,\phi \vDash C$
- Completeness:
    - If a G and ϕ exists such that $G,\phi \vDash C$, then the solver returns it

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)
- Semantics describes when a constraint is satisfied by a model

How are these related?

- Soundness
  - If the solver returns <G, ϕ>, then $G,\phi \models C$
- Completeness:
  - If a G and ϕ exists such that $G,\phi \models C$, then the solver returns it
  - If no such G or ϕ exists, the solver fails

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)
- Semantics describes when a constraint is satisfied by a model

How are these related?

- Soundness
  - If the solver returns <G, ϕ>, then $G,\phi \vDash C$
- Completeness:
  - If a G and ϕ exists such that $G,\phi \vDash C$, then the solver returns it
  - If no such G or ϕ exists, the solver fails
- Principality

Semantics vs Algorithm

What is the difference?

- Algorithm computes a solution (= model)
- Semantics describes when a constraint is satisfied by a model

How are these related?

- Soundness
  - If the solver returns <G, ϕ>, then $G,\phi \vDash C$
- Completeness:
  - If a G and ϕ exists such that $G,\phi \vDash C$, then the solver returns it
  - If no such G or ϕ exists, the solver fails
- Principality
  - The solver finds the most general ϕ

Term Equality & Unification

Syntactic Terms

Generic Terms

terms	t, u
functions	f, g, h

Syntactic Terms

Generic Terms

```
terms      t, u
functions  f, g, h
```

```
INT()
FUN(INT(),INT())
```

Syntactic Terms

Generic Terms

```
terms      t, u
functions  f, g, h
```

```
INT()
FUN(INT(),INT())
```

Syntactic Terms

Generic Terms

```
terms      t, u
functions  f, g, h
```

```
INT()
FUN(INT(),INT())
```

⇨

$f(t_0,\ldots,t_n)$

Syntactic Terms

Generic Terms

```
terms      t, u
functions  f, g, h
```

function symbol

```
INT()
FUN(INT(),INT())
```

⇨

$f(t_0,\dots,t_n)$

Syntactic Terms

Generic Terms

```
terms      t, u
functions  f, g, h
```

```
INT()
FUN(INT(),INT())
```

$\Rightarrow$ $\underbrace{f}_{\text{function symbol}}(\underbrace{t_0,\ldots,t_n}_{\text{arguments}})$

Syntactic Terms

Generic Terms

terms	t, u
functions	f, g, h

```
INT()
FUN(INT(),INT())
```

⇒

function symbol

arguments

$f(t_0, \ldots, t_n)$

arity

Syntactic Terms

Generic Terms

terms	t, u
functions	f, g, h

```
INT()
FUN(INT(),INT())
```

function symbol

arguments

$f(t_0,\dots,t_n)$

arity

Syntactic Equality

```
f(t0,…,tn) == g(u0,…,um) if
- f = g, and n = m
- ti == ui for every i
```

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

```
f(g(),a)
```

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

variable

f(g(),a)

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

variable

`f(g(),a)`

`ϕ = { a -> f(g(),b), b -> h() }`

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

variable

f(g(),a)

substitution

ϕ = { a -> f(g(),b), b -> h() }

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

variable

f(g(),a)

substitution

ϕ = { a -> f(g(),b), b -> h() }

domain

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

variable

$f(g(),a)$

substitution

$\phi = \{ a \text{ -> } f(g(),b),\ b \text{ -> } h() \}$

domain

$$
\begin{aligned}
\phi(a) &= t && \text{if } \{ a \text{ -> } t \} \text{ in } \phi \\
\phi(a) &= a && \text{otherwise} \\
\phi(f(t_0,\dots,t_n)) &= f(\phi(t_0),\dots,\phi(t_n)) &&
\end{aligned}
$$

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

variable

`f(g(),a)`

substitution

`ϕ = { a -> f(g(),b), b -> h() }`

domain

```
ϕ(a)            = t                        if { a -> t } in ϕ
ϕ(a)            = a                        otherwise
ϕ(f(t0,…,tn)) = f(ϕ(t0),…,ϕ(tn))
```

`f(g(),f(g(),b))`

Variables and Substitution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

variable

f(g(),a)

substitution

ϕ = { a -> f(g(),b), b -> h() }

domain

$$\phi(a) = t \quad \text{if } \{ a \to t \} \text{ in } \phi$$
$$\phi(a) = a \quad \text{otherwise}$$
$$\phi(f(t_0,\ldots,t_n)) = f(\phi(t_0),\ldots,\phi(t_n))$$

f(g(),f(g(),b))

<u>ground term</u>: a term without variables

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

unifier: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

`f(a,g()) == f(h(),b)` ⇨

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

unifier: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()
                           b -> g()
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇒   a -> h()
                           b -> g()   ⇒
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()   ⇨   f(h(),g()) == f(h(),g())
                           b -> g()
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()    ⇨   f(h(),g()) == f(h(),g())
                           b -> g()
```

```
g(a,f(b)) == g(f(h()),a)
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()    ⇨   f(h(),g()) == f(h(),g())
                           b -> g()
```

```
g(a,f(b)) == g(f(h()),a)   ⇨
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()    ⇨   f(h(),g()) == f(h(),g())
                           b -> g()
```

```
g(a,f(b)) == g(f(h()),a)   ⇨   a -> f(h())
                               b -> h()
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()    ⇨   f(h(),g()) == f(h(),g())
                           b -> g()

g(a,f(b)) == g(f(h()),a)   ⇨   a -> f(h())   ⇨
                               b -> h()
```

Unifiers

```
terms        t, u
functions    f, g, h
variables    a, b, c
substitution ϕ
```

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()     ⇨   f(h(),g()) == f(h(),g())
                           b -> g()

g(a,f(b)) == g(f(h()),a)   ⇨   a -> f(h())   ⇨   g(f(h()),f(h())) == g(f(h()),f(h()))
                               b -> h()
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()       ⇨   f(h(),g()) == f(h(),g())
                           b -> g()

g(a,f(b)) == g(f(h()),a)   ⇨   a -> f(h())   ⇨   g(f(h()),f(h())) == g(f(h()),f(h()))
                               b -> h()

f(a,h()) == g(h(),b)
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

__unifier__: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   ⇨   a -> h()        ⇨   f(h(),g()) == f(h(),g())
                           b -> g()

g(a,f(b)) == g(f(h()),a)   ⇨   a -> f(h())   ⇨   g(f(h()),f(h())) == g(f(h()),f(h()))
                               b -> h()

f(a,h()) == g(h(),b)   ⇨
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   =>   a -> h()     =>   f(h(),g()) == f(h(),g())
                            b -> g()

g(a,f(b)) == g(f(h()),a)   =>   a -> f(h())   =>   g(f(h()),f(h())) == g(f(h()),f(h()))
                                b -> h()

f(a,h()) == g(h(),b)   =>   no unifier, f != g
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   =>   a -> h()    =>   f(h(),g()) == f(h(),g())
                            b -> g()

g(a,f(b)) == g(f(h()),a)   =>   a -> f(h())   =>   g(f(h()),f(h())) == g(f(h()),f(h()))
                                b -> h()

f(a,h()) == g(h(),b)   =>   no unifier, f != g

f(b,b) == b
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)  =>  a -> h()       =>  f(h(),g()) == f(h(),g())
                          b -> g()

g(a,f(b)) == g(f(h()),a)  =>  a -> f(h())  =>  g(f(h()),f(h())) == g(f(h()),f(h()))
                              b -> h()

f(a,h()) == g(h(),b)  =>  no unifier, f != g

f(b,b) == b  =>
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   =>   a -> h()     =>   f(h(),g()) == f(h(),g())
                            b -> g()

g(a,f(b)) == g(f(h()),a)   =>   a -> f(h())   =>   g(f(h()),f(h())) == g(f(h()),f(h()))
                                b -> h()

f(a,h()) == g(h(),b)   =>   no unifier, f != g

f(b,b) == b   =>   b -> f(b,b)
```

Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

<u>unifier</u>: a substitution that makes terms equal

```
f(a,g()) == f(h(),b)   =>   a -> h()       =>   f(h(),g()) == f(h(),g())
                            b -> g()

g(a,f(b)) == g(f(h()),a)   =>   a -> f(h())   =>   g(f(h()),f(h())) == g(f(h()),f(h()))
                                b -> h()

f(a,h()) == g(h(),b)   =>   no unifier, f != g

f(b,b) == b   =>   b -> f(b,b)   <--   not idempotent
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

`f(a,b) == f(b,c)` ⇒

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

```
f(a,b) == f(b,c)
```

⇒

```
a -> b
c -> b
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

```
f(a,b) == f(b,c)
```

⇒

```
a -> b
c -> b
```

⇒

```
f(b,b) == f(b,b)
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

```
a -> g()
b -> g()
c -> g()
```

```
f(a,b) == f(b,c)
```

```
a -> b
c -> b
```

```
f(b,b) == f(b,b)
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

```
a -> g()
b -> g()      =>   f(g(),g()) == f(g(),g())
c -> g()
```

```
f(a,b) == f(b,c)   =>   a -> b
                        c -> b   =>   f(b,b) == f(b,b)
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

```
f(a,b) == f(b,c)
```

```
a -> g()
b -> g()
c -> g()
```

```
f(g(),g()) == f(g(),g())
```

```
a -> b
c -> b
```

```
f(b,b) == f(b,b)
```

```
b -> a
c -> a
```

Most General Unifiers

```
terms        t, u
functions    f, g, h
variables    a, b, c
substitution ϕ
```

```
f(a,b) == f(b,c)
```

```
a -> g()
b -> g()
c -> g()
```

```
f(g(),g()) == f(g(),g())
```

```
a -> b
c -> b
```

```
f(b,b) == f(b,b)
```

```
b -> a
c -> a
```

```
f(a,a) == f(a,a)
```

Most General Unifiers

```
terms         t, u
functions     f, g, h
variables     a, b, c
substitution  ϕ
```

```
f(a,b) == f(b,c)
```

```
a -> g()
b -> g()
c -> g()
```

```
f(g(),g()) == f(g(),g())
```

```
a -> b
c -> b
```

```
f(b,b) == f(b,b)
```

```
b -> a
c -> a
```

```
f(a,a) == f(a,a)
```

most general unifiers

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

every unifier is an instance of a most general unifier

```
a -> b
b -> b
c -> b
```

```
a -> g()
b -> g()
c -> g()
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

every unifier is an instance of a most general unifier

(implicit) identity case

```
a -> b
b -> b
c -> b
```

```
a -> g()
b -> g()
c -> g()
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

every unifier is an instance of a most general unifier

(implicit) identity case

```
a -> b
b -> b
c -> b
```

b -> g()

```
a -> g()
b -> g()
c -> g()
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

every unifier is an instance of a most general unifier

(implicit) identity case

```
a -> b
b -> b
c -> b
```

b -> g()

```
a -> g()
b -> g()
c -> g()
```

most general unifiers are related by renaming substitutions

```
a -> b
b -> b
c -> b
```

```
a -> a
b -> a
c -> a
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

every unifier is an instance of a most general unifier

(implicit) identity case

```
a -> b
b -> b
c -> b
```

b -> g()

```
a -> g()
b -> g()
c -> g()
```

most general unifiers are related by renaming substitutions

```
a -> b
b -> b
c -> b
```

b -> a

```
a -> a
b -> a
c -> a
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

every unifier is an instance of a most general unifier

(implicit) identity case

```
a -> b
b -> b
c -> b
```

b -> g()

```
a -> g()
b -> g()
c -> g()
```

most general unifiers are related by renaming substitutions

```
a -> b
b -> b
c -> b
```

b -> a

```
a -> a
b -> a
c -> a
```

```
a -> a
b -> a
c -> a
```

```
a -> b
b -> b
c -> b
```

Most General Unifiers

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

every unifier is an instance of a most general unifier

(implicit) identity case

```
a -> b                          a -> g()
b -> b       b -> g()           b -> g()
c -> b      ========>           c -> g()
```

most general unifiers are related by renaming substitutions

```
a -> b                          a -> a
b -> b       b -> a             b -> a
c -> b      ========>           c -> a
```

```
a -> a                          a -> b
b -> a       a -> b             b -> b
c -> a      ========>           c -> b
```

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t₀,...,tₙ) and u == g(u₀,...,uₘ):
    if f == g and n == m:
      for i := 1 to n:
        unify(tᵢ, uᵢ)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t0,...,tn) and u == g(u0,...,um):
    if f == g and n == m:
      for i := 1 to n:
        unify(ti, ui)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

t == a
instantiate variable

```
terms         t, u
functions     f, g, h
variables     a, b, c
substitution  ϕ
```

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t0,...,tn) and u == g(u0,...,um):
    if f == g and n == m:
      for i := 1 to n:
        unify(ti, ui)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

t == a
instantiate variable

u == b
instantiate variable

```
terms          t, u
functions      f, g, h
variables      a, b, c
substitution   ϕ
```

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t₀,...,tₙ) and u == g(u₀,...,uₘ):
    if f == g and n == m:
      for i := 1 to n:
        unify(tᵢ, uᵢ)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

t == a
instantiate variable
u == b
instantiate variable
b == b
equal variables

```
terms           t, u
functions       f, g, h
variables       a, b, c
substitution ϕ
```

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t₀,...,tₙ) and u == g(u₀,...,uₘ):
    if f == g and n == m:
      for i := 1 to n:
        unify(tᵢ, uᵢ)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

$t == a$
instantiate variable

$u == b$
instantiate variable

$b == b$
equal variables

$t == f(t_0,\dots,t_5)$, $u == f(u_0,\dots,u_5)$
matching terms

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t₀,...,tₙ) and u == g(u₀,...,uₘ):
    if f == g and n == m:
      for i := 1 to n:
        unify(tᵢ, uᵢ)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

t == a
instantiate variable

u == b
instantiate variable

b == b
equal variables

$t == f(t_0,\ldots,t_5)$, $u == f(u_0,\ldots,u_5)$
matching terms

$t == f(t_0,\ldots,t_5)$, $u == g(u_0,\ldots,u_3)$
mismatching terms

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t0,...,tn) and u == g(u0,...,um):
    if f == g and n == m:
      for i := 1 to n:
        unify(ti, ui)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$t == a$
instantiate variable

$u == b$
instantiate variable

$b == b$
equal variables

$t == f(t_0,\dots,t_5)$, $u == f(u_0,\dots,u_5)$
matching terms

$t == f(t_0,\dots,t_5)$, $u == g(u_0,\dots,u_3)$
mismatching terms

$t == f(t_0,\dots,t_5)$, $u == b$
swap terms

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t0,...,tn) and u == g(u0,...,um):
    if f == g and n == m:
      for i := 1 to n:
        unify(ti, ui)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

$t == a$
instantiate variable

$u == b$
instantiate variable

$b == b$
equal variables

$t == f(t_0,...,t_5)$, $u == f(u_0,...,u_5)$
matching terms

$t == f(t_0,...,t_5)$, $u == g(u_0,...,u_3)$
mismatching terms

$t == f(t_0,...,t_5)$, $u == b$
swap terms

$t == a$, $u == k(g(a,f()))$
recursive terms

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

Unification

```
global ϕ
def unify(t, u):
  if t is a variable:
    t := ϕ(t)
  if u is a variable:
    u := ϕ(u)
  if t is a variable and t == u:
    pass
  else if t == f(t₀,...,tₙ) and u == g(u₀,...,uₘ):
    if f == g and n == m:
      for i := 1 to n:
        unify(tᵢ, uᵢ)
    else:
      fail "different function symbols"
  else if t is not a variable:
    unify(u, t)
  else if t occurs in u:
    fail "recursive term"
  else:
    ϕ += { t -> u }
```

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$t == a$
instantiate variable

$u == b$
instantiate variable

$b == b$
equal variables

$t == f(t_0,\ldots,t_5)$, $u == f(u_0,\ldots,u_5)$
matching terms

$t == f(t_0,\ldots,t_5)$, $u == g(u_0,\ldots,u_3)$
mismatching terms

$t == f(t_0,\ldots,t_5)$, $u == b$
swap terms

$t == a$, $u == k(g(a,f()))$
recursive terms

$t == a$, $u == k(u_0,\ldots,u_5)$
extend unifier

Properties of Unification

Properties of Unification

Soundness

Properties of Unification

Soundness

- If the algorithm returns a unifier, it makes the terms equal

Properties of Unification

Soundness

– If the algorithm returns a unifier, it makes the terms equal

Completeness

Properties of Unification

Soundness

– If the algorithm returns a unifier, it makes the terms equal

Completeness

– If a unifier exists, the algorithm will return it

Properties of Unification

Soundness

– If the algorithm returns a unifier, it makes the terms equal

Completeness

– If a unifier exists, the algorithm will return it

Principality

Properties of Unification

Soundness

– If the algorithm returns a unifier, it makes the terms equal

Completeness

– If a unifier exists, the algorithm will return it

Principality

– If the algorithm returns a unifier, it is a most general unifier

Properties of Unification

Soundness

– If the algorithm returns a unifier, it makes the terms equal

Completeness

– If a unifier exists, the algorithm will return it

Principality

– If the algorithm returns a unifier, it is a most general unifier

Termination

Properties of Unification

Soundness

– If the algorithm returns a unifier, it makes the terms equal

Completeness

– If a unifier exists, the algorithm will return it

Principality

– If the algorithm returns a unifier, it is a most general unifier

Termination

– The algorithm always returns a unifier or fails

Efficient Unification with Union-Find

Complexity of Unification

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$$

Complexity of Unification

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

```
h(a₁        , …,aₙ              , f(b₀,b₀), …, f(bₙ₋₁,bₙ₋₁), aₙ) ==
h(f(a₀,a₀), …,f(aₙ₋₁,aₙ₋₁), b₁,          …, bₙ₋₁          , bₙ)
```

⇩

```
a₁ -> f(a₀,a₀)
a₂ -> f(f(a₀,a₀), f(a₀,a₀))
aᵢ -> … 2^(i+1)-1 subterms …
b₁ -> f(a₀,a₀)
b₂ -> f(f(a₀,a₀), f(a₀,a₀))
bᵢ -> … 2^(i+1)-1 subterms …
```

Complexity of Unification

Space complexity

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$$

a_1 -> $f(a_0,a_0)$
a_2 -> $f(f(a_0,a_0), f(a_0,a_0))$
a_i -> … $2^{i+1}-1$ subterms …
b_1 -> $f(a_0,a_0)$
b_2 -> $f(f(a_0,a_0), f(a_0,a_0))$
b_i -> … $2^{i+1}-1$ subterms …

Complexity of Unification

Space complexity

- Exponential

```
terms          t, u
functions      f, g, h
variables      a, b, c
substitution   ϕ
```

```
h(a₁        , …,aₙ              , f(b₀,b₀), …, f(bₙ₋₁,bₙ₋₁), aₙ) ==
h(f(a₀,a₀), …,f(aₙ₋₁,aₙ₋₁), b₁,          …, bₙ₋₁          , bₙ)
```

⇩

```
a₁ -> f(a₀,a₀)
a₂ -> f(f(a₀,a₀), f(a₀,a₀))
aᵢ -> … 2ⁱ⁺¹-1 subterms …
b₁ -> f(a₀,a₀)
b₂ -> f(f(a₀,a₀), f(a₀,a₀))
bᵢ -> … 2ⁱ⁺¹-1 subterms …
```

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$h(a_1, \ldots, a_n, f(b_0, b_0), \ldots, f(b_{n-1}, b_{n-1}), a_n) ==$
$h(f(a_0, a_0), \ldots, f(a_{n-1}, a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$

$a_1 \to f(a_0, a_0)$
$a_2 \to f(f(a_0, a_0), f(a_0, a_0))$
$a_i \to \ldots\ 2^{i+1}-1$ subterms $\ldots$
$b_1 \to f(a_0, a_0)$
$b_2 \to f(f(a_0, a_0), f(a_0, a_0))$
$b_i \to \ldots\ 2^{i+1}-1$ subterms $\ldots$

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

```
terms        t, u
functions    f, g, h
variables    a, b, c
substitution ϕ
```

```
h(a₁          , …,aₙ              , f(b₀,b₀), …, f(bₙ₋₁,bₙ₋₁), aₙ) ==
h(f(a₀,a₀), …,f(aₙ₋₁,aₙ₋₁), b₁,        …, bₙ₋₁          , bₙ)
```

↓

```
a₁ -> f(a₀,a₀)
a₂ -> f(f(a₀,a₀), f(a₀,a₀))
aᵢ -> … 2^(i+1)-1 subterms …
b₁ -> f(a₀,a₀)
b₂ -> f(f(a₀,a₀), f(a₀,a₀))
bᵢ -> … 2^(i+1)-1 subterms …
```

fully applied

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$$

a_1 -> $f(a_0,a_0)$
a_2 -> $f(f(a_0,a_0), f(a_0,a_0))$
a_i -> … $2^{i+1}-1$ subterms …
b_1 -> $f(a_0,a_0)$
b_2 -> $f(f(a_0,a_0), f(a_0,a_0))$
b_i -> … $2^{i+1}-1$ subterms …

fully applied

triangular

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$$

$a_1 \to f(a_0,a_0)$
$a_2 \to f(f(a_0,a_0), f(a_0,a_0))$
$a_i \to \ldots\ 2^{i+1}-1$ subterms …
$b_1 \to f(a_0,a_0)$
$b_2 \to f(f(a_0,a_0), f(a_0,a_0))$
$b_i \to \ldots\ 2^{i+1}-1$ subterms …

fully applied

$a_1 \to f(a_0,a_0)$
$a_2 \to f(a_1,a_1)$
$a_i \to \ldots$ 3 subterms …
$b_1 \to f(a_0,a_0)$
$b_2 \to f(a_1,a_1)$
$b_i \to \ldots$ 3 subterms …

triangular

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

Time complexity

```
terms          t, u
functions      f, g, h
variables      a, b, c
substitution   ϕ
```

$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) ==$
$h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$

$a_1 \to f(a_0,a_0)$
$a_2 \to f(f(a_0,a_0), f(a_0,a_0))$
$a_i \to \ldots\ 2^{i+1}-1$ subterms …
$b_1 \to f(a_0,a_0)$
$b_2 \to f(f(a_0,a_0), f(a_0,a_0))$
$b_i \to \ldots\ 2^{i+1}-1$ subterms …

fully applied

$a_1 \to f(a_0,a_0)$
$a_2 \to f(a_1,a_1)$
$a_i \to \ldots$ 3 subterms …
$b_1 \to f(a_0,a_0)$
$b_2 \to f(a_1,a_1)$
$b_i \to \ldots$ 3 subterms …

triangular

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

Time complexity

- Exponential

```
terms        t, u
functions    f, g, h
variables    a, b, c
substitution ϕ
```

```
h(a1       , …,an          , f(b0,b0), …, f(bn-1,bn-1), an) ==
h(f(a0,a0), …,f(an-1,an-1), b1,        …, bn-1        , bn)
```

```
a1 -> f(a0,a0)
a2 -> f(f(a0,a0), f(a0,a0))
ai -> … 2^(i+1)-1 subterms …
b1 -> f(a0,a0)
b2 -> f(f(a0,a0), f(a0,a0))
bi -> … 2^(i+1)-1 subterms …
```

fully applied

```
a1 -> f(a0,a0)
a2 -> f(a1,a1)
ai -> … 3 subterms …
b1 -> f(a0,a0)
b2 -> f(a1,a1)
bi -> … 3 subterms …
```

triangular

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

Time complexity

- Exponential
- Recursive calls on terms

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)$$

$a_1 \to f(a_0,a_0)$
$a_2 \to f(f(a_0,a_0), f(a_0,a_0))$
$a_i \to \ldots\ 2^{i+1}-1$ subterms …
$b_1 \to f(a_0,a_0)$
$b_2 \to f(f(a_0,a_0), f(a_0,a_0))$
$b_i \to \ldots\ 2^{i+1}-1$ subterms …

fully applied

$a_1 \to f(a_0,a_0)$
$a_2 \to f(a_1,a_1)$
$a_i \to \ldots$ 3 subterms …
$b_1 \to f(a_0,a_0)$
$b_2 \to f(a_1,a_1)$
$b_i \to \ldots$ 3 subterms …

triangular

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

Time complexity

- Exponential
- Recursive calls on terms

Solution

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$
\begin{aligned}
&h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) == \\
&h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)
\end{aligned}
$$

$$
\begin{aligned}
a_1 &\to f(a_0,a_0) \\
a_2 &\to f(f(a_0,a_0), f(a_0,a_0)) \\
a_i &\to \ldots\ 2^{i+1}-1 \text{ subterms } \ldots \\
b_1 &\to f(a_0,a_0) \\
b_2 &\to f(f(a_0,a_0), f(a_0,a_0)) \\
b_i &\to \ldots\ 2^{i+1}-1 \text{ subterms } \ldots
\end{aligned}
$$

fully applied

$$
\begin{aligned}
a_1 &\to f(a_0,a_0) \\
a_2 &\to f(a_1,a_1) \\
a_i &\to \ldots\ 3 \text{ subterms } \ldots \\
b_1 &\to f(a_0,a_0) \\
b_2 &\to f(a_1,a_1) \\
b_i &\to \ldots\ 3 \text{ subterms } \ldots
\end{aligned}
$$

triangular

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

Time complexity

- Exponential
- Recursive calls on terms

Solution

- Union-Find algorithm

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$\begin{aligned}&h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) == \\ &h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)\end{aligned}$$

a_1 -> $f(a_0,a_0)$
a_2 -> $f(f(a_0,a_0), f(a_0,a_0))$
a_i -> … $2^{i+1}-1$ subterms …
b_1 -> $f(a_0,a_0)$
b_2 -> $f(f(a_0,a_0), f(a_0,a_0))$
b_i -> … $2^{i+1}-1$ subterms …

fully applied

a_1 -> $f(a_0,a_0)$
a_2 -> $f(a_1,a_1)$
a_i -> … 3 subterms …
b_1 -> $f(a_0,a_0)$
b_2 -> $f(a_1,a_1)$
b_i -> … 3 subterms …

triangular

Complexity of Unification

Space complexity

- Exponential
- Representation of unifier

Time complexity

- Exponential
- Recursive calls on terms

Solution

- Union-Find algorithm
- Complexity growth can be considered constant

terms	t, u
functions	f, g, h
variables	a, b, c
substitution	ϕ

$$h(a_1, \dots, a_n, f(b_0,b_0), \dots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \dots, f(a_{n-1},a_{n-1}), b_1, \dots, b_{n-1}, b_n)$$

$a_1 \rightarrow f(a_0,a_0)$
$a_2 \rightarrow f(f(a_0,a_0), f(a_0,a_0))$
$a_i \rightarrow \dots\ 2^{i+1}-1$ subterms …
$b_1 \rightarrow f(a_0,a_0)$
$b_2 \rightarrow f(f(a_0,a_0), f(a_0,a_0))$
$b_i \rightarrow \dots\ 2^{i+1}-1$ subterms …

fully applied

$a_1 \rightarrow f(a_0,a_0)$
$a_2 \rightarrow f(a_1,a_1)$
$a_i \rightarrow \dots$ 3 subterms …
$b_1 \rightarrow f(a_0,a_0)$
$b_2 \rightarrow f(a_1,a_1)$
$b_i \rightarrow \dots$ 3 subterms …

triangular

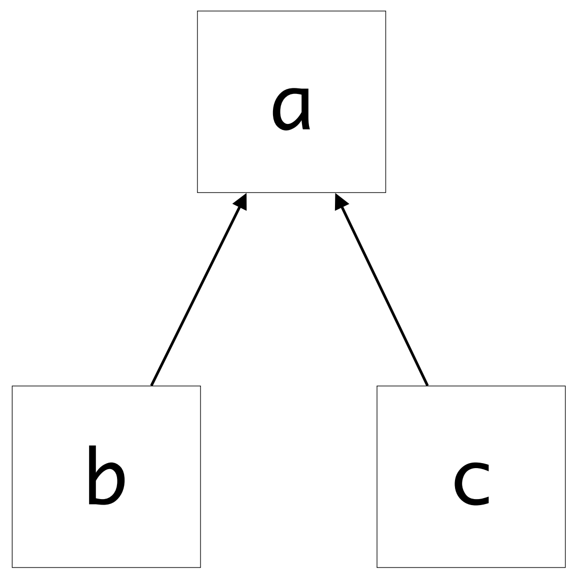
Set Representatives

Set Representatives

a == b
c == a

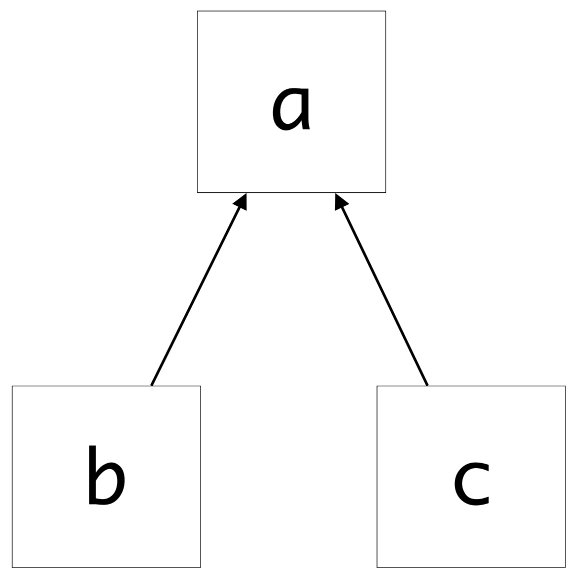
Set Representatives

a == b
c == a



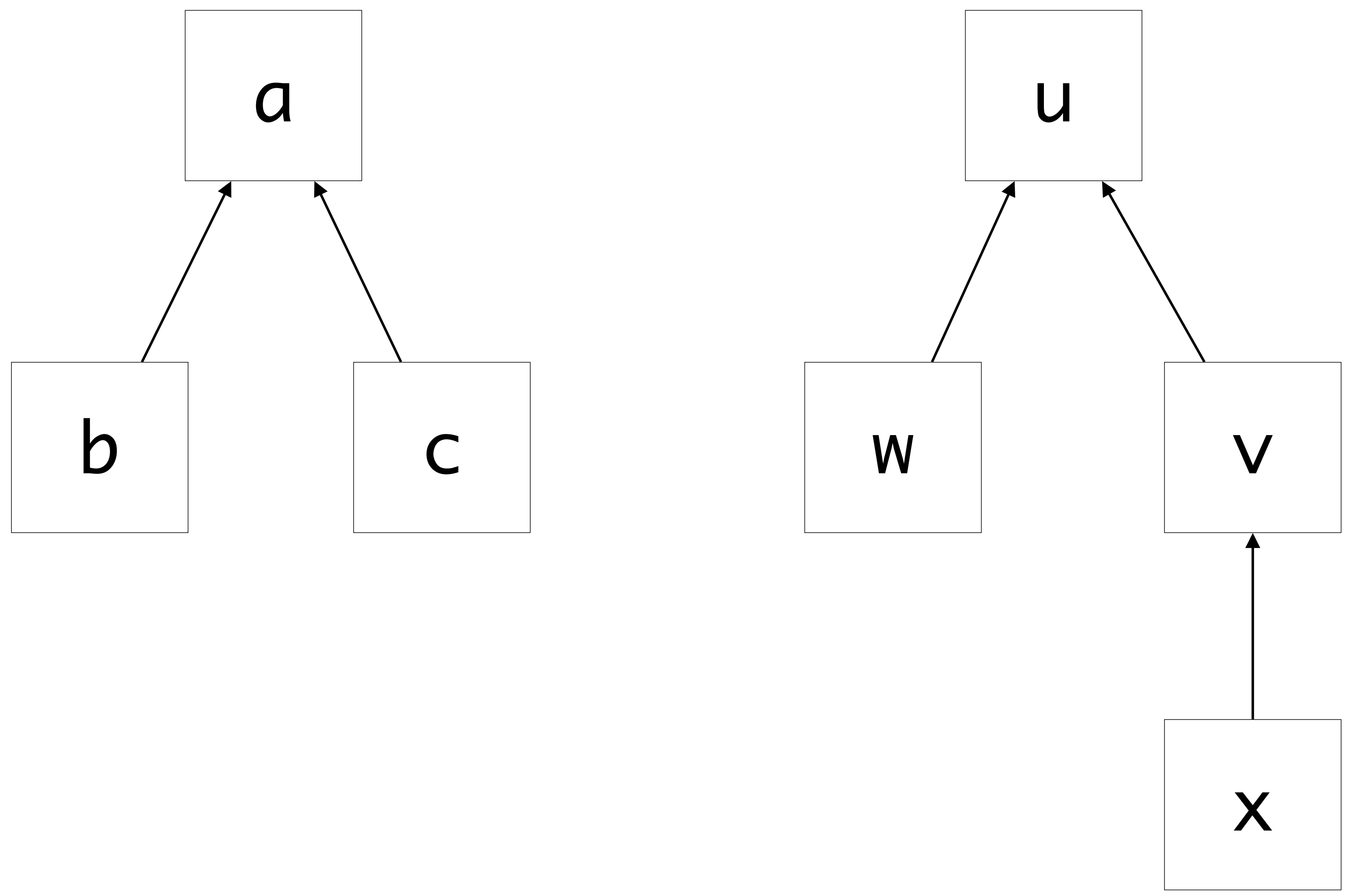
Set Representatives

a == b
c == a
u == w
v == u
x == v



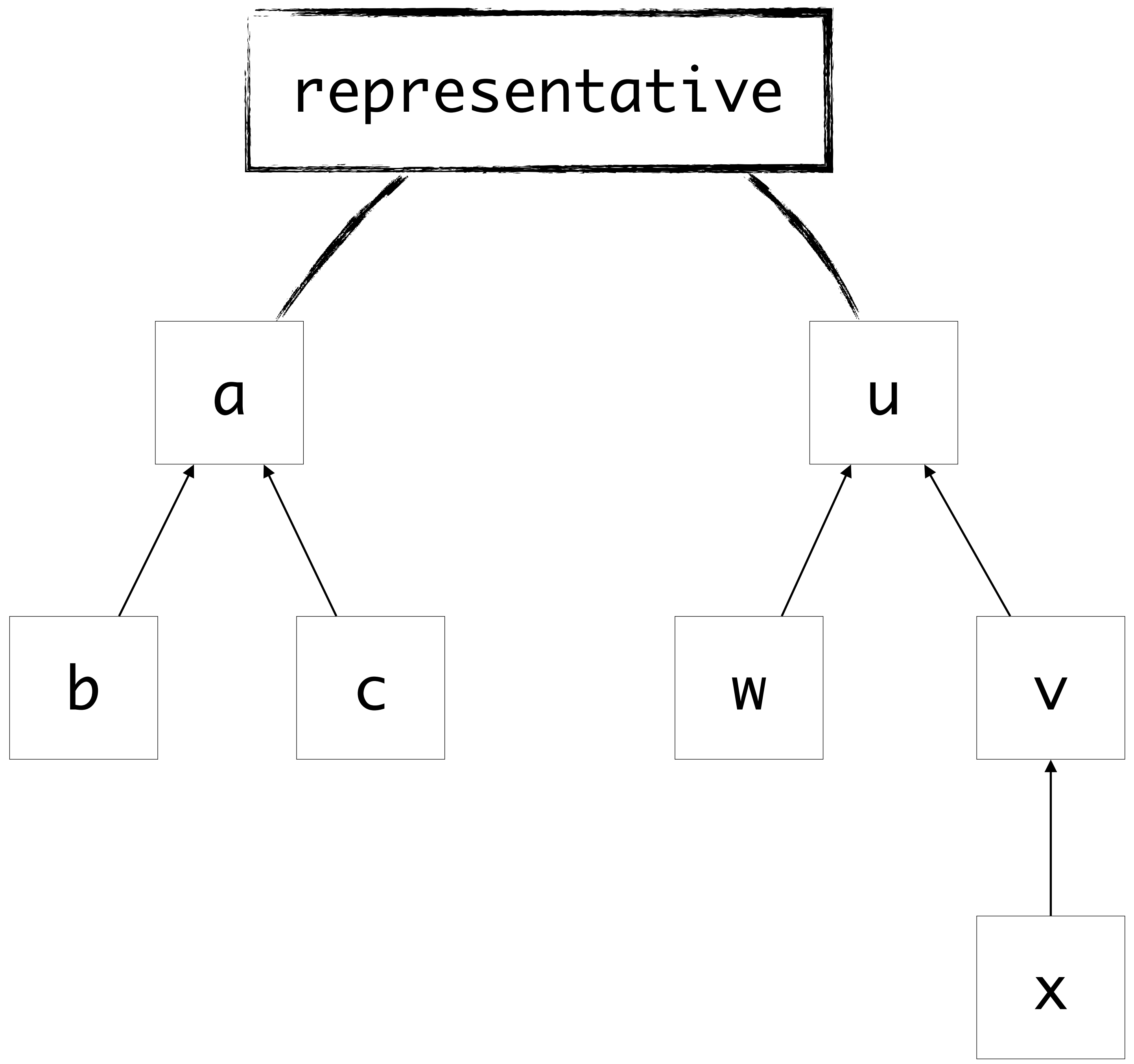
Set Representatives

a == b
c == a
u == w
v == u
x == v



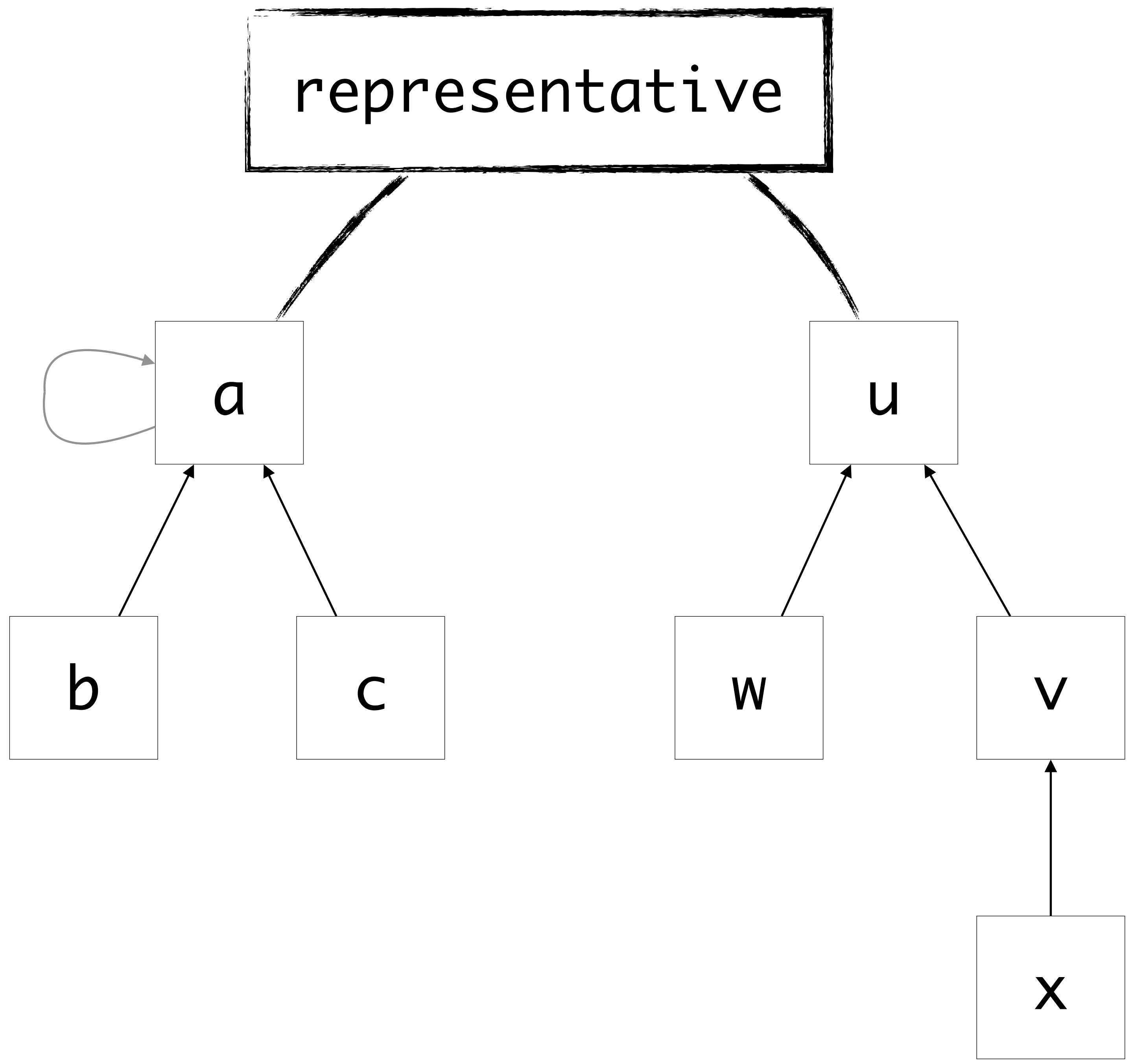
Set Representatives

a == b
c == a
u == w
v == u
x == v



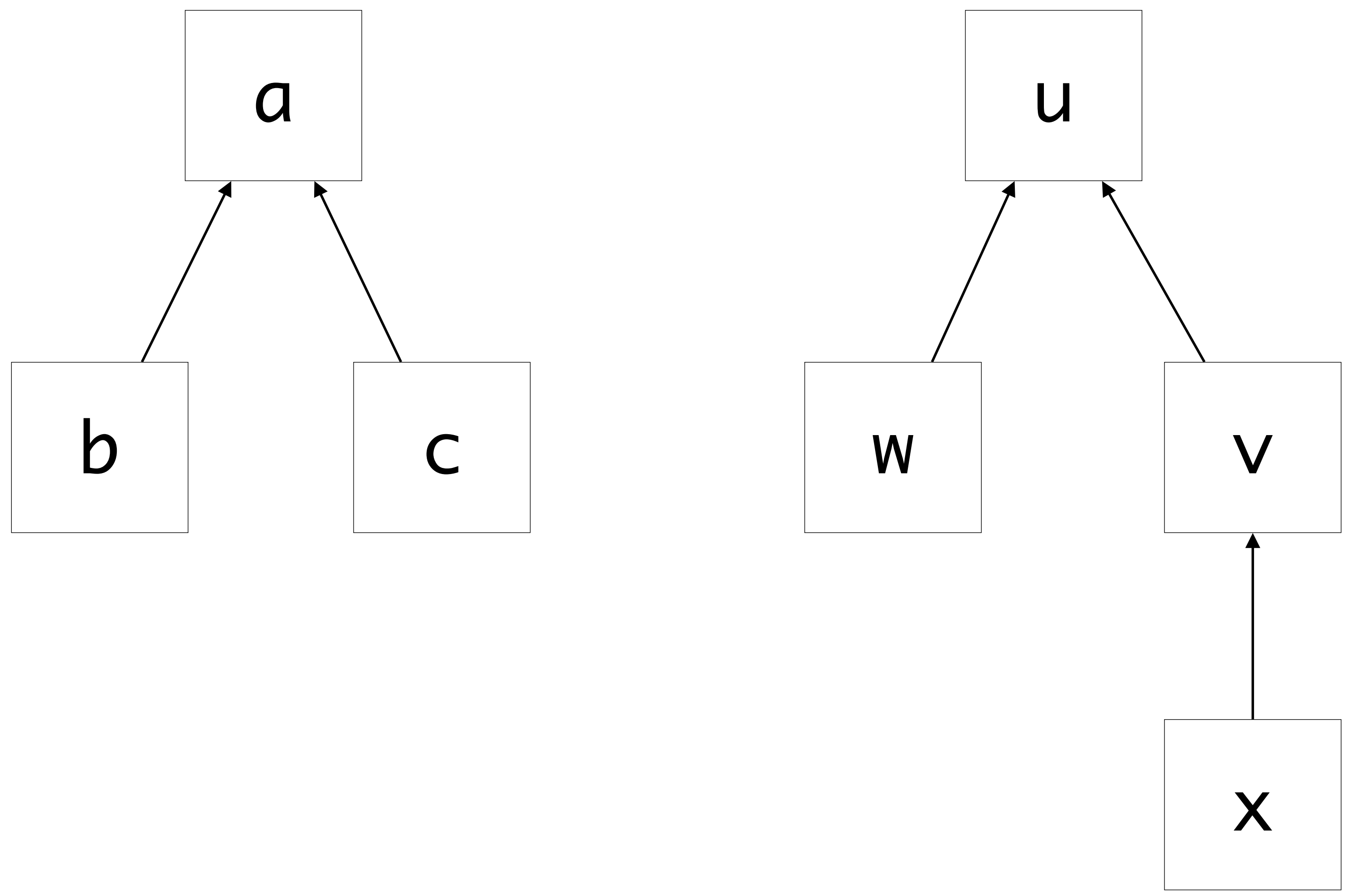
Set Representatives

a == b
c == a
u == w
v == u
x == v



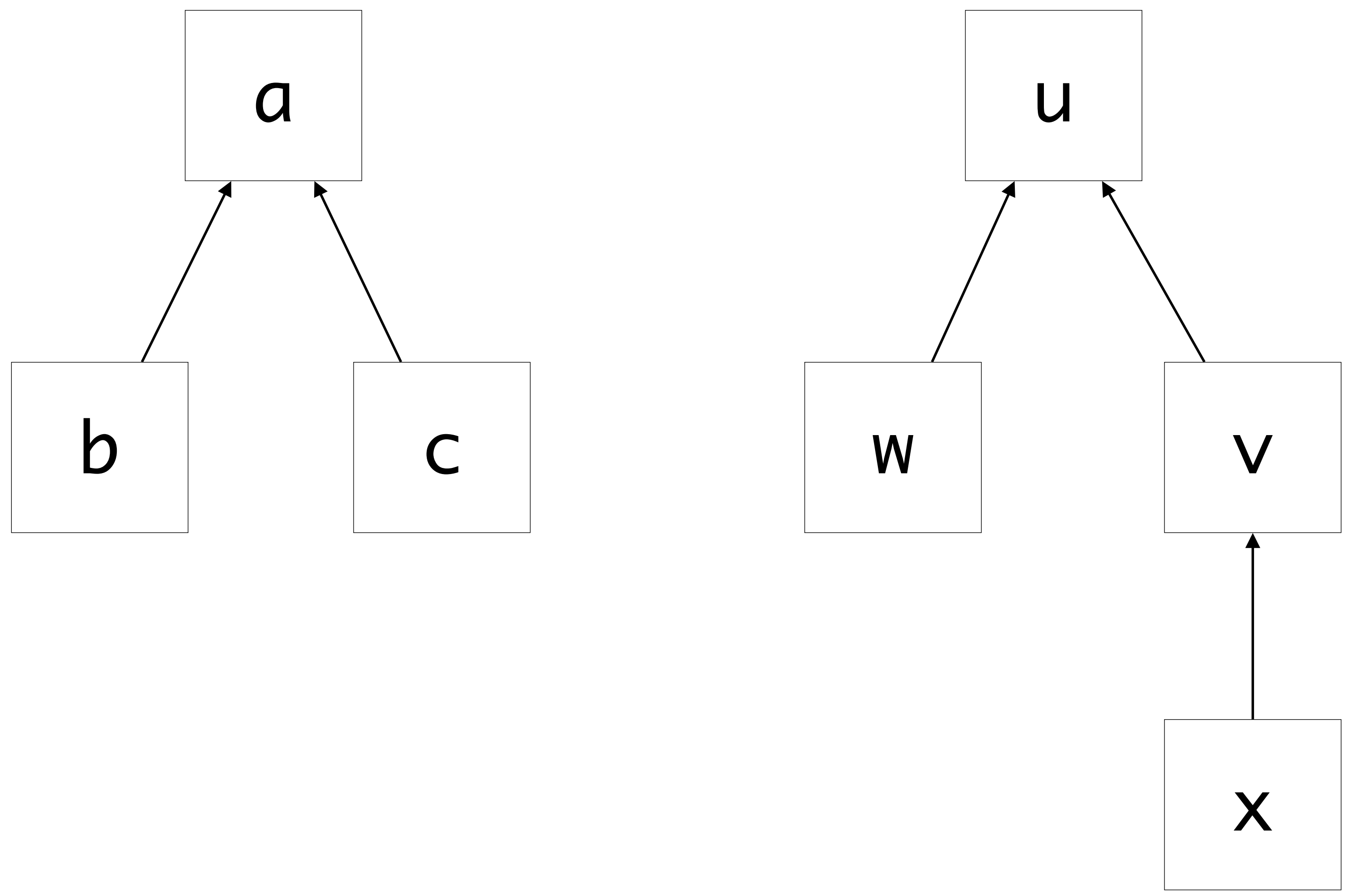
Set Representatives

a == b
c == a
u == w
v == u
x == v



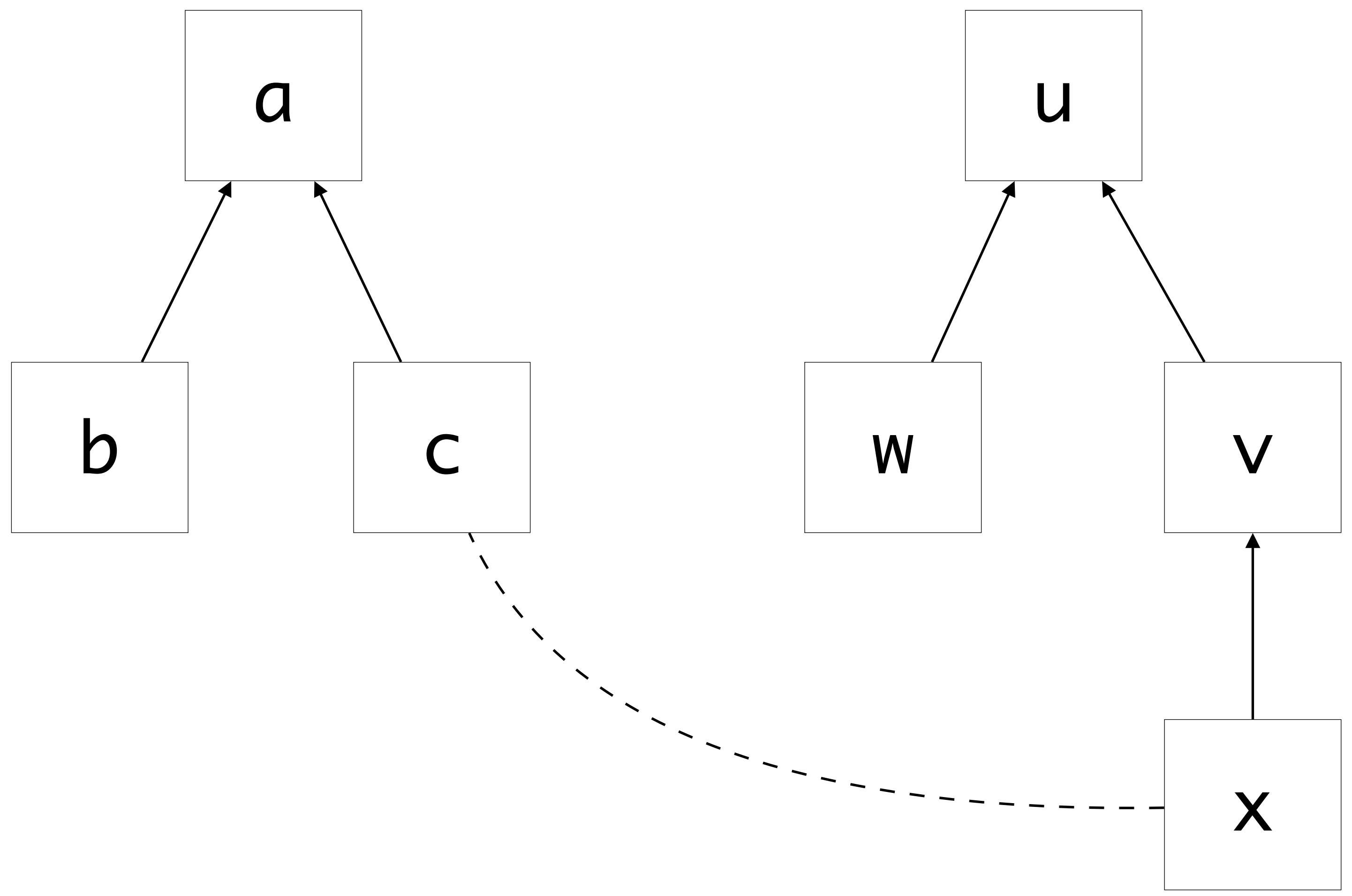
Set Representatives

a == b
c == a
u == w
v == u
x == v
x == c



Set Representatives

a == b
c == a
u == w
v == u
x == v
x == c



Set Representatives

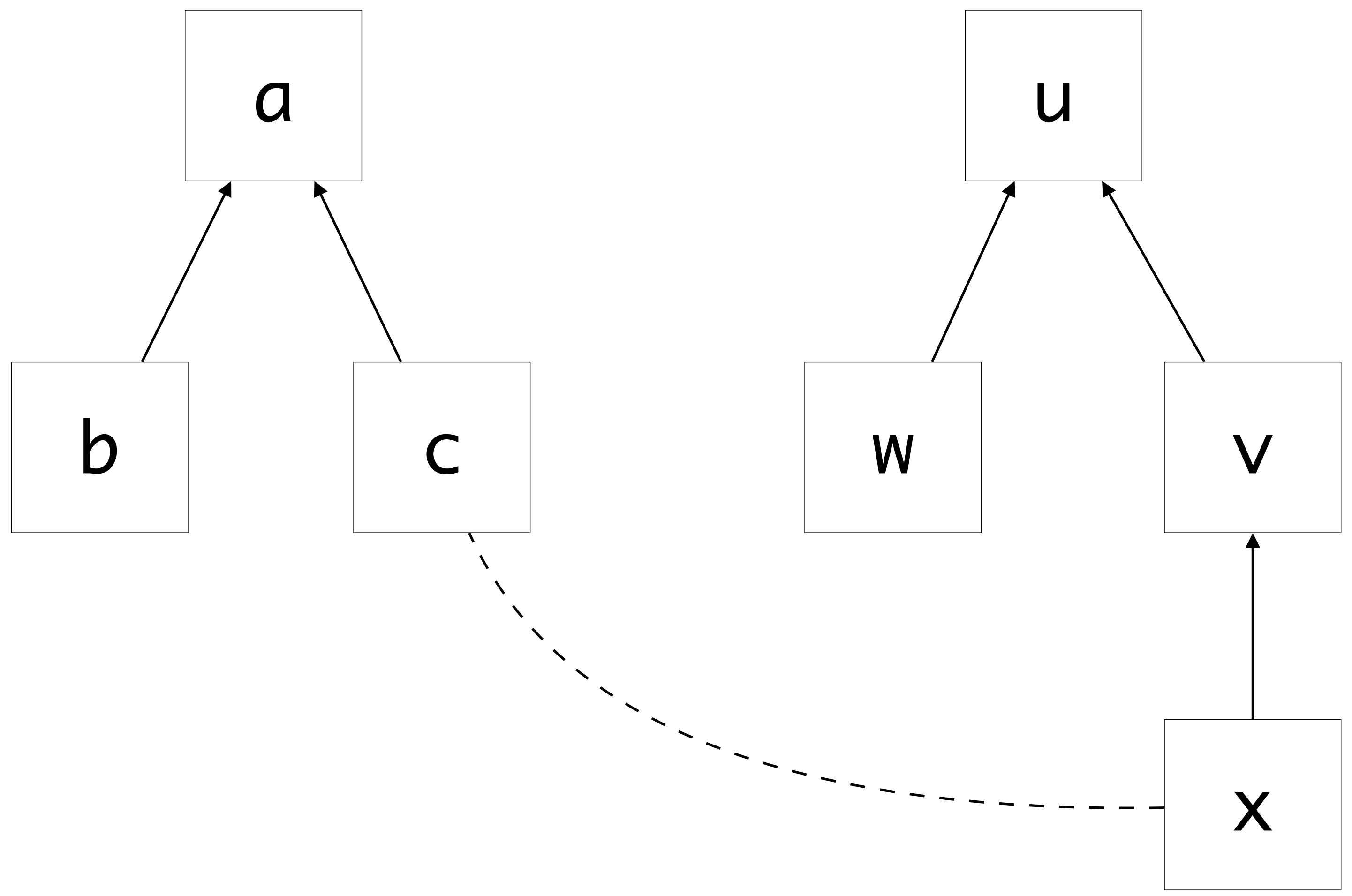
```
FIND(a):
  b := rep(a)
  if b == a:
     return a
  else
     return FIND(b)



UNION(a1,a2):
  b1 := FIND(a1)
  b2 := FIND(a2)
  LINK(b1,b2)

LINK(a1,a2):
  rep(a1) := a2
```

a == b
c == a
u == w
v == u
x == v
x == c



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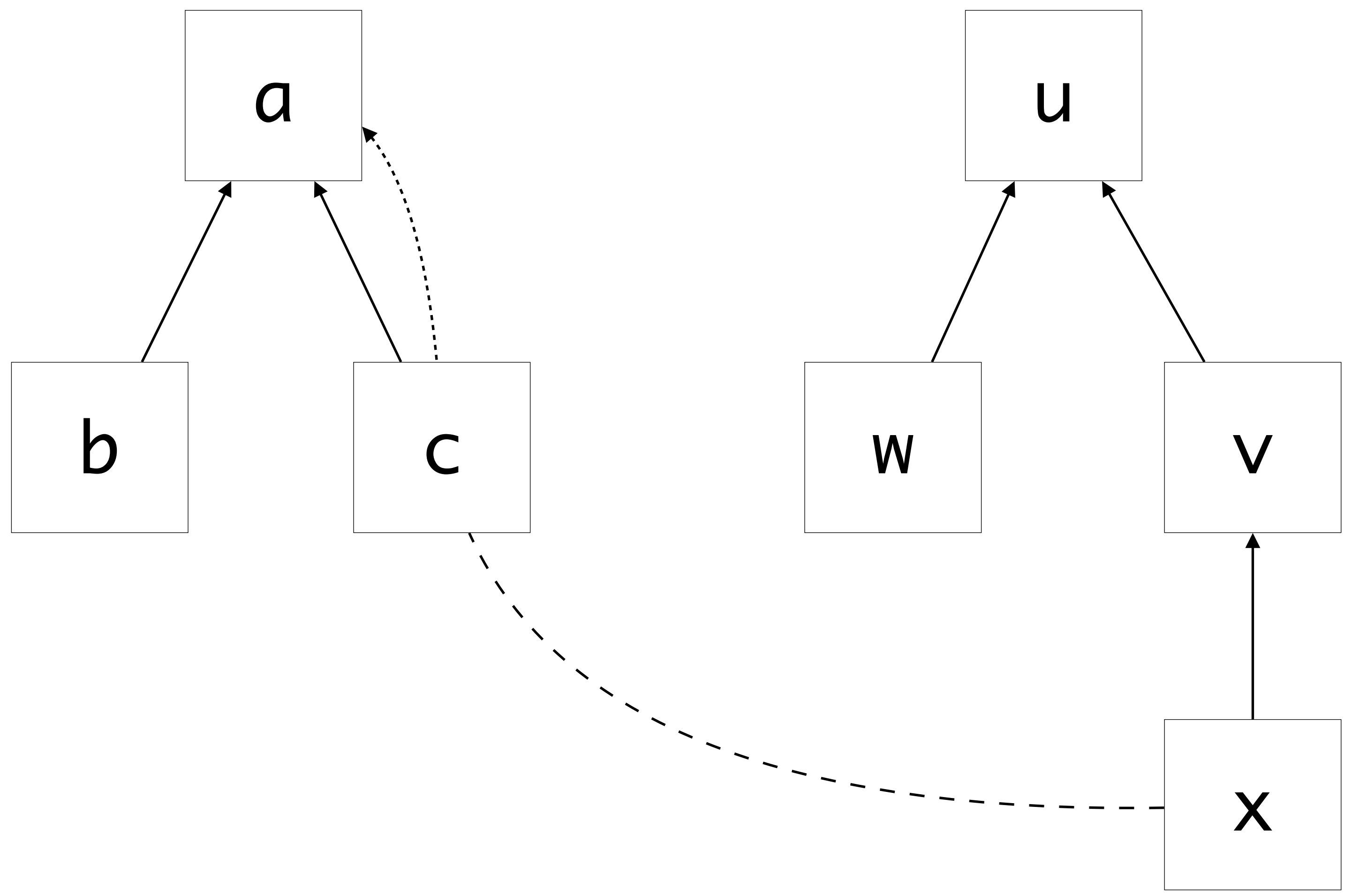

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```
FIND(a):
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     return a
  else
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```

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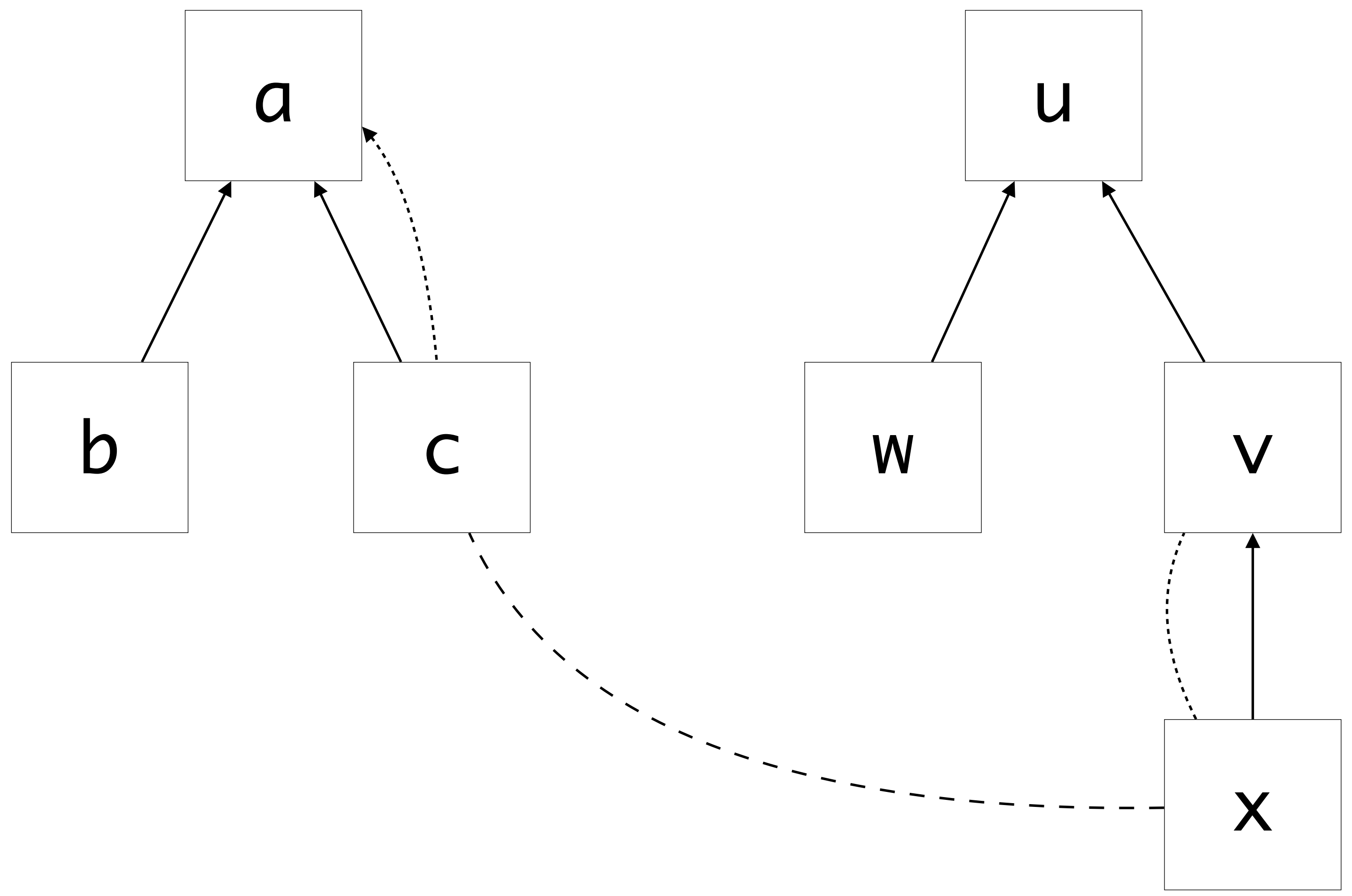

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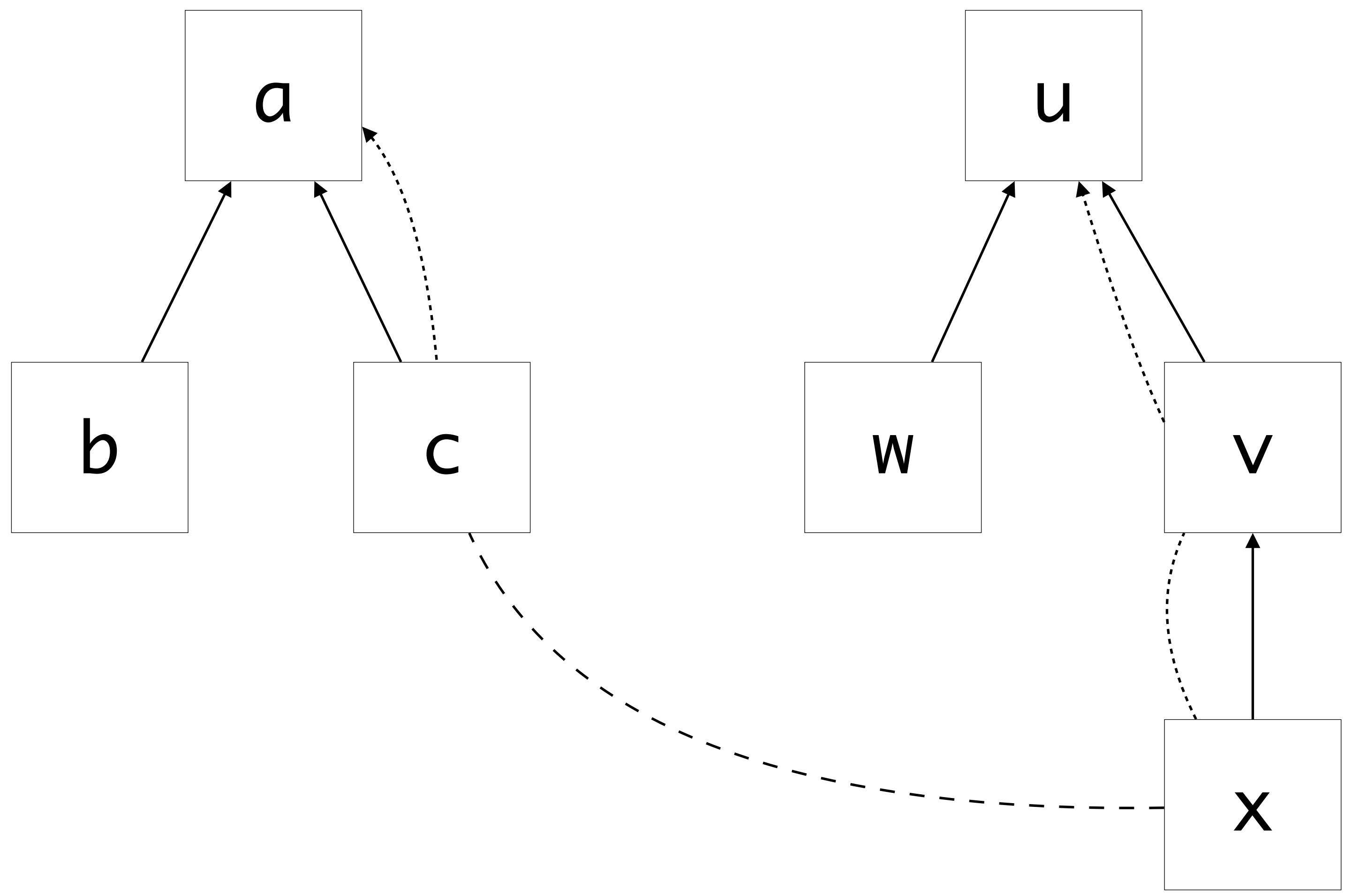

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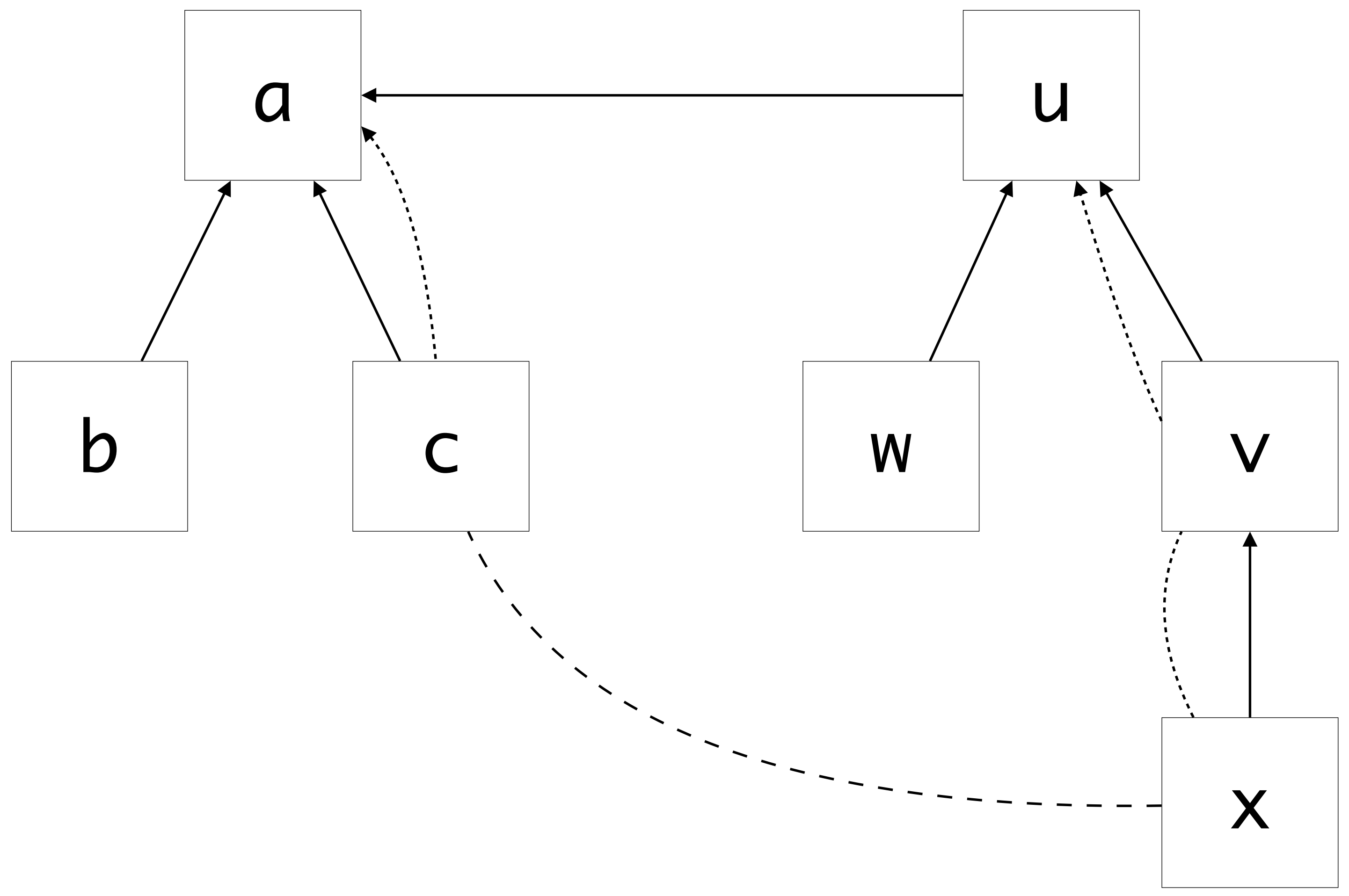

Path Compression

```
FIND(a):
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```

```
 …
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x == c
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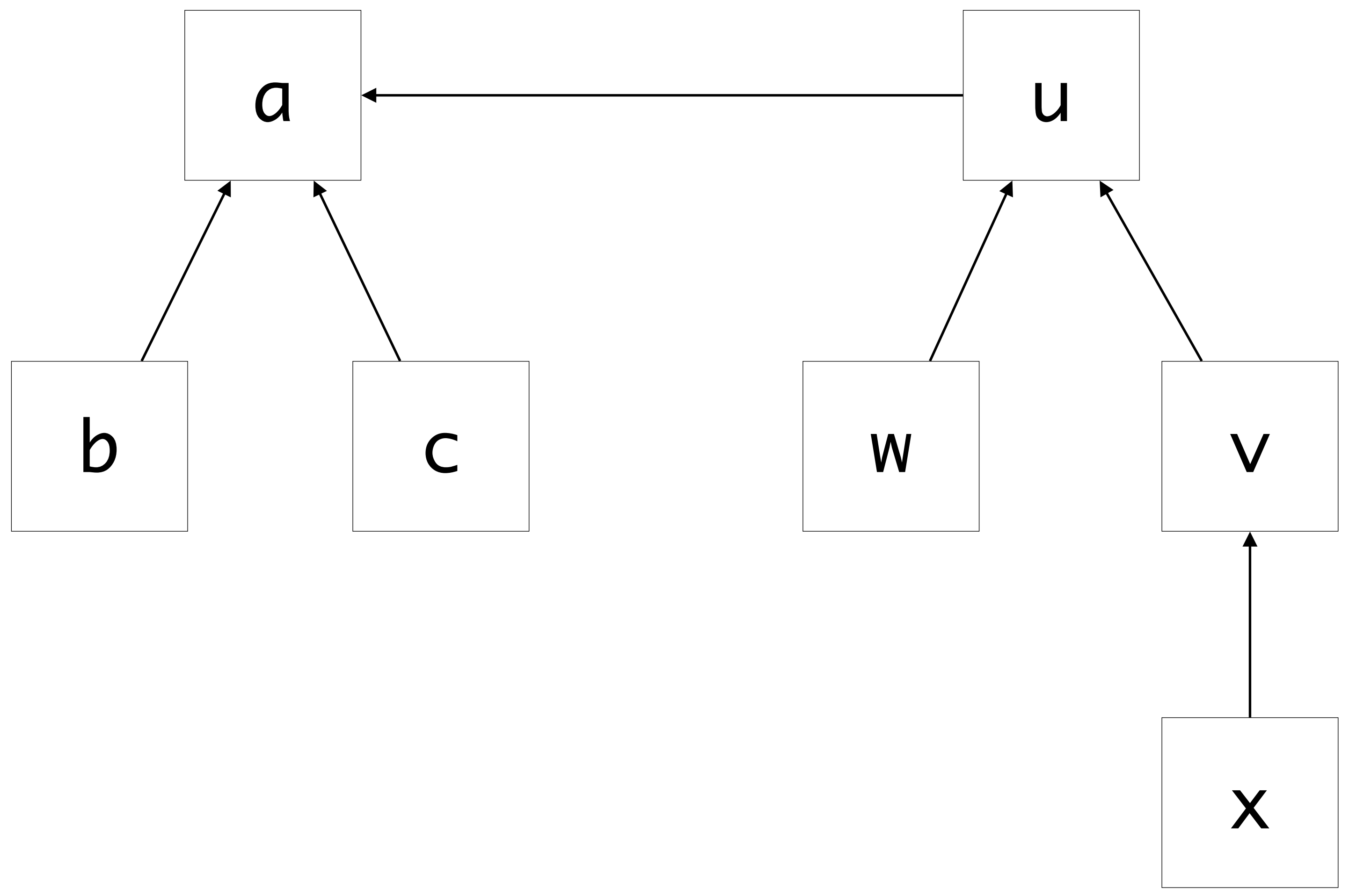
```



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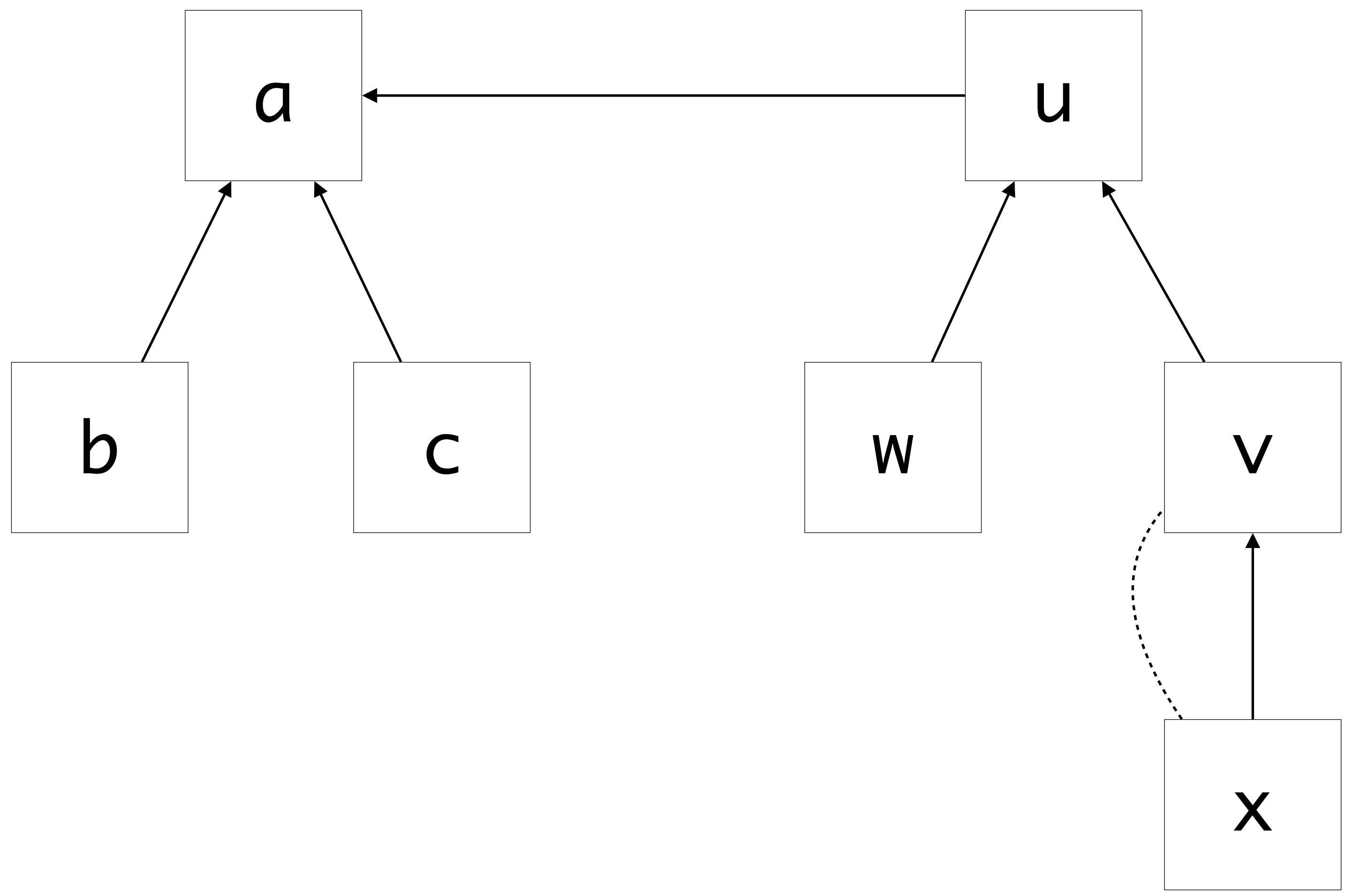

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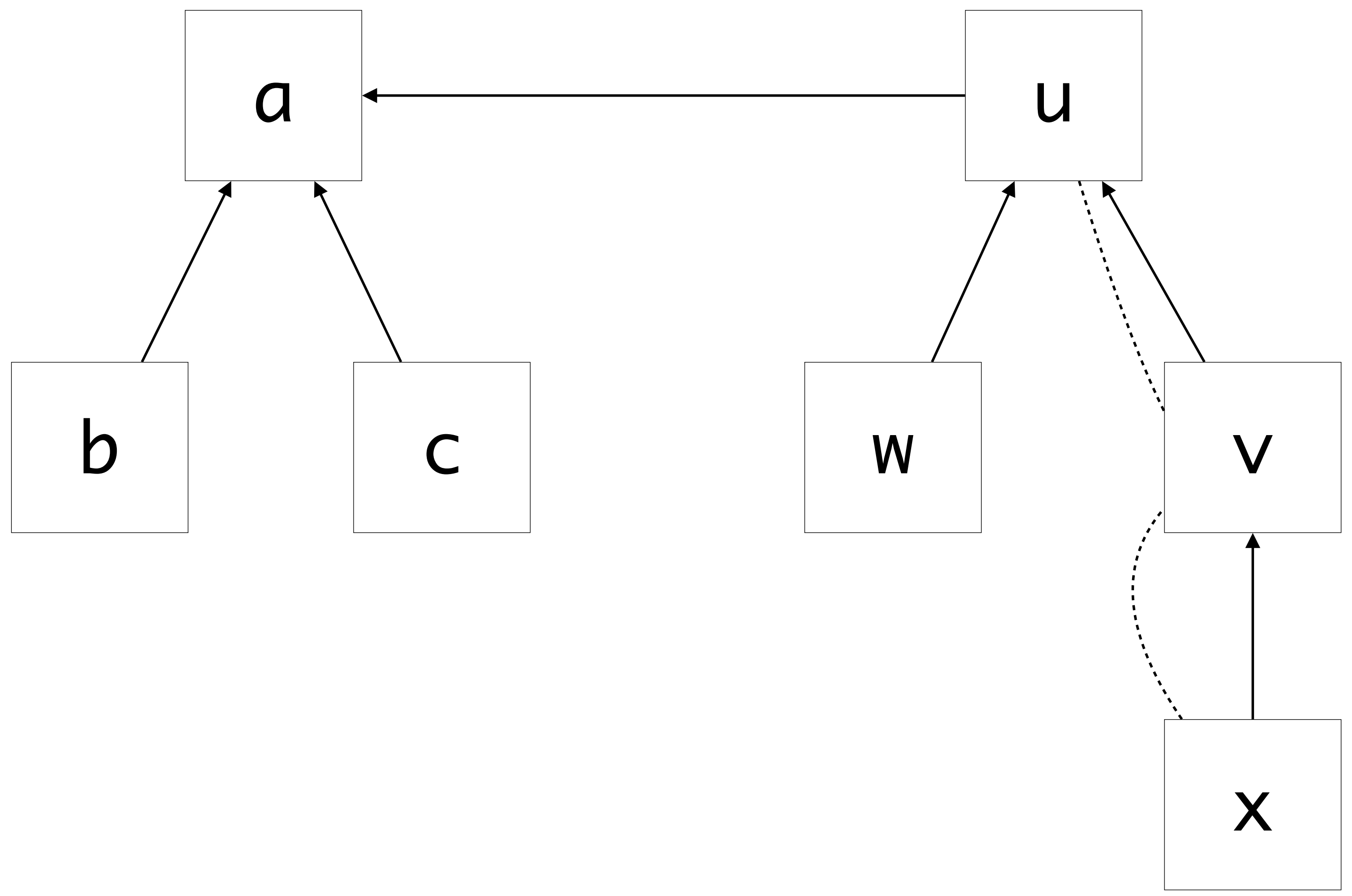

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```

…
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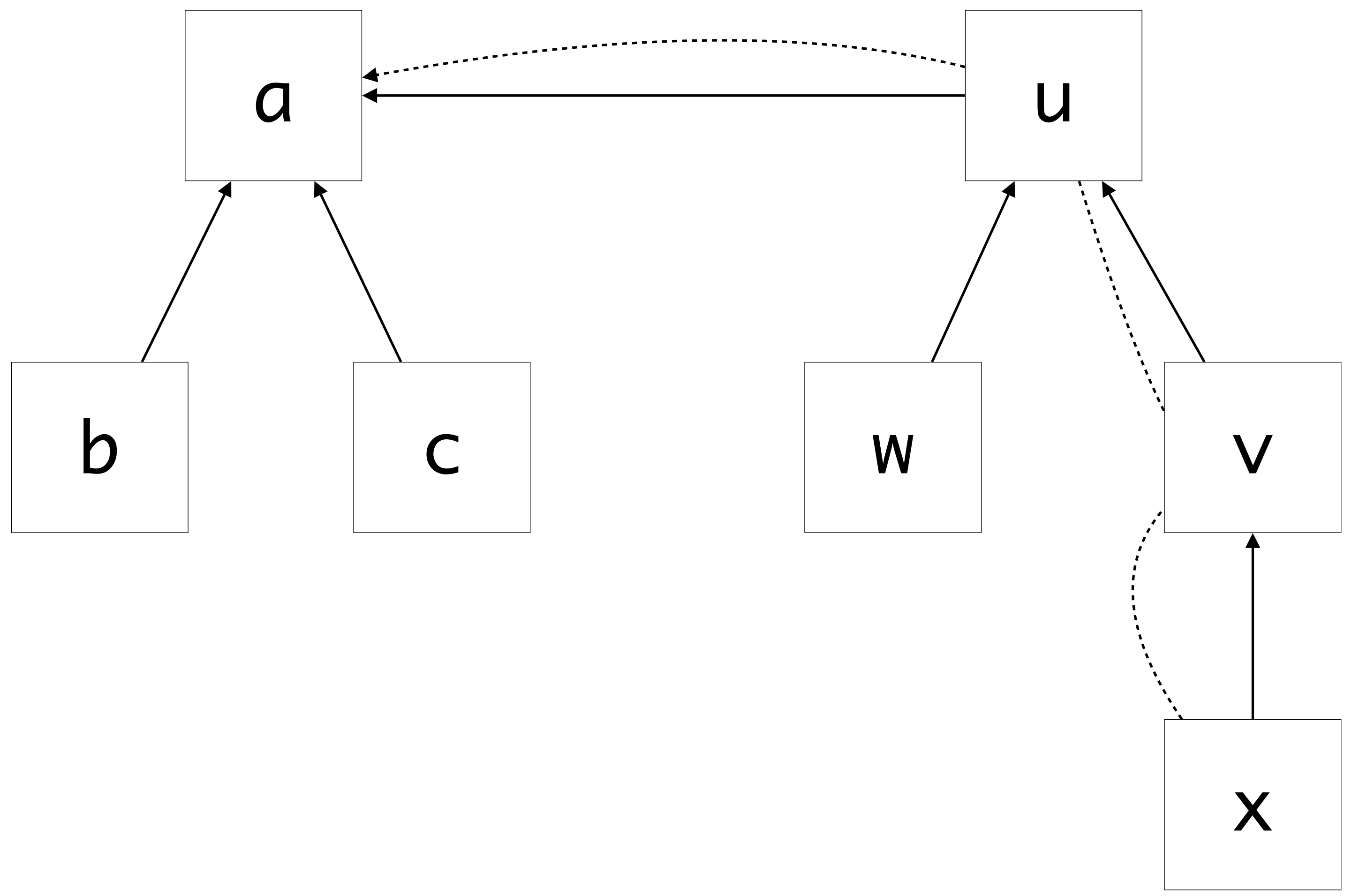

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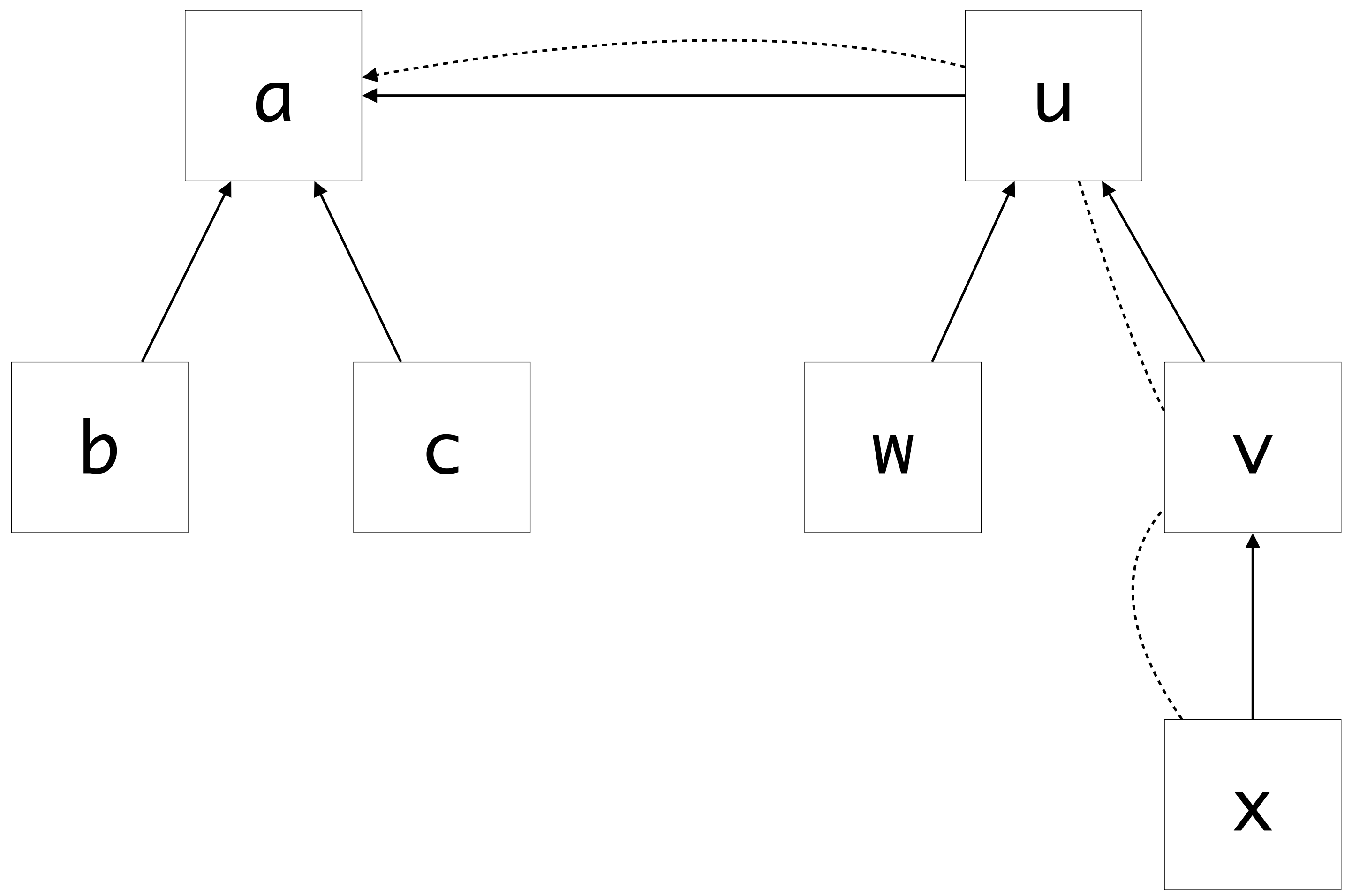

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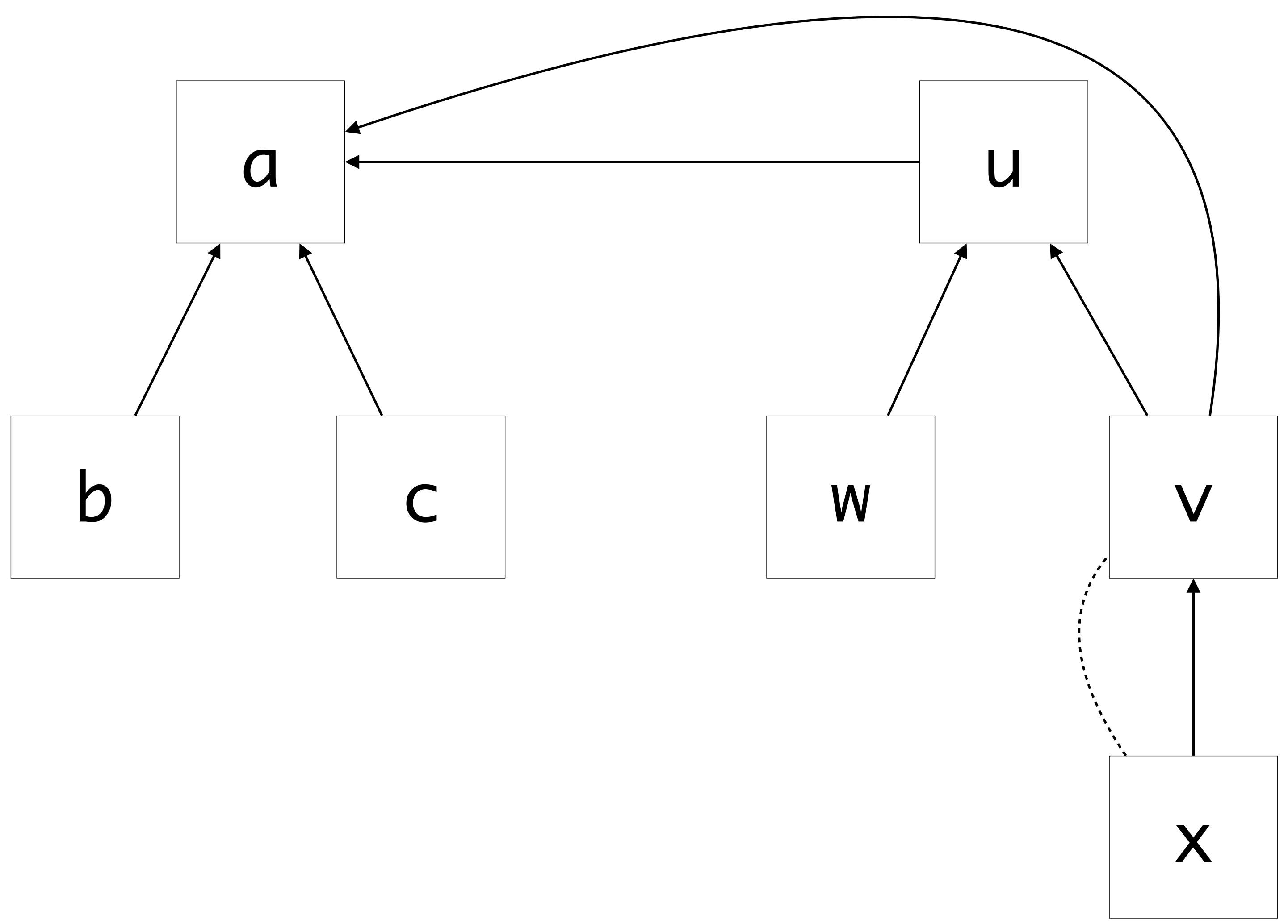

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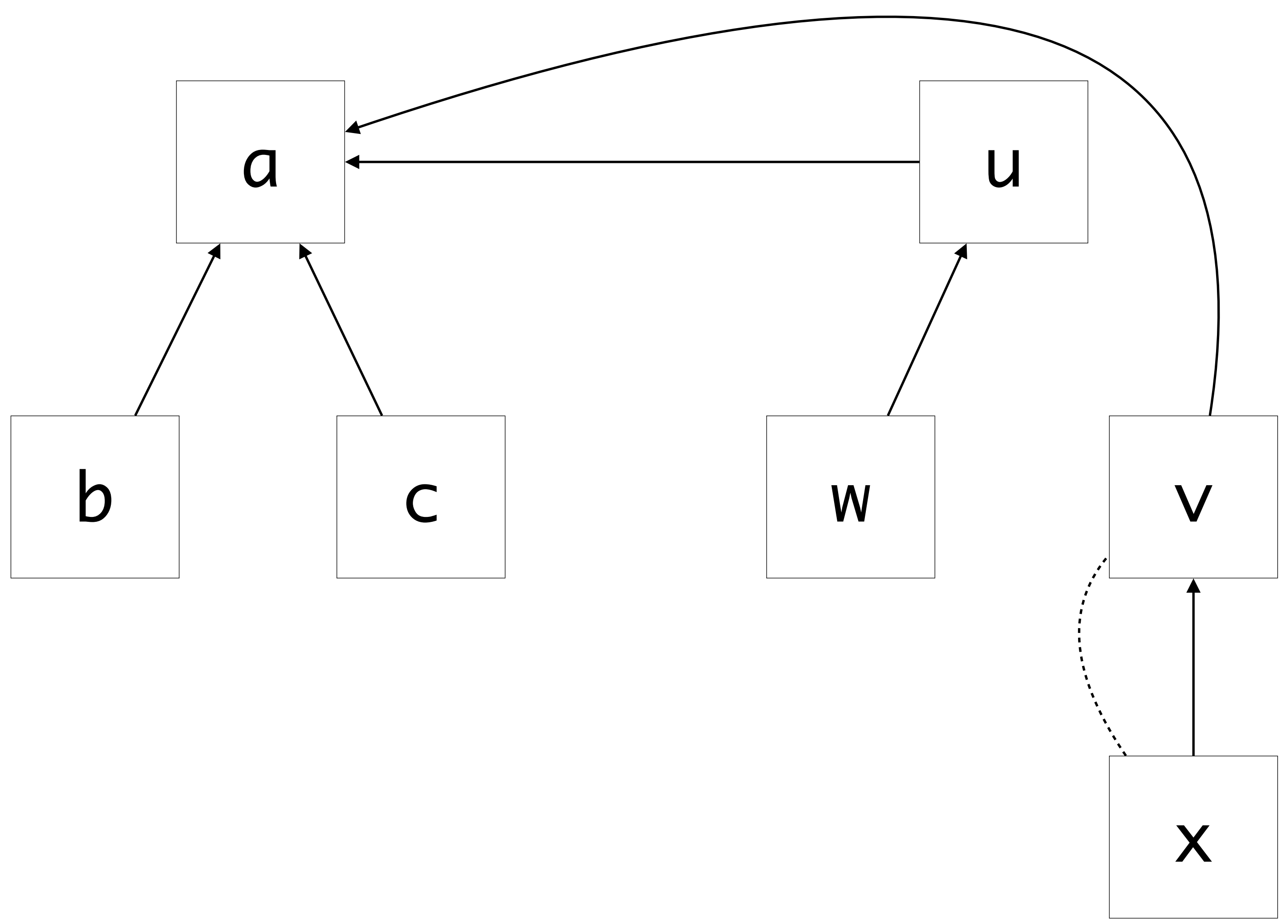

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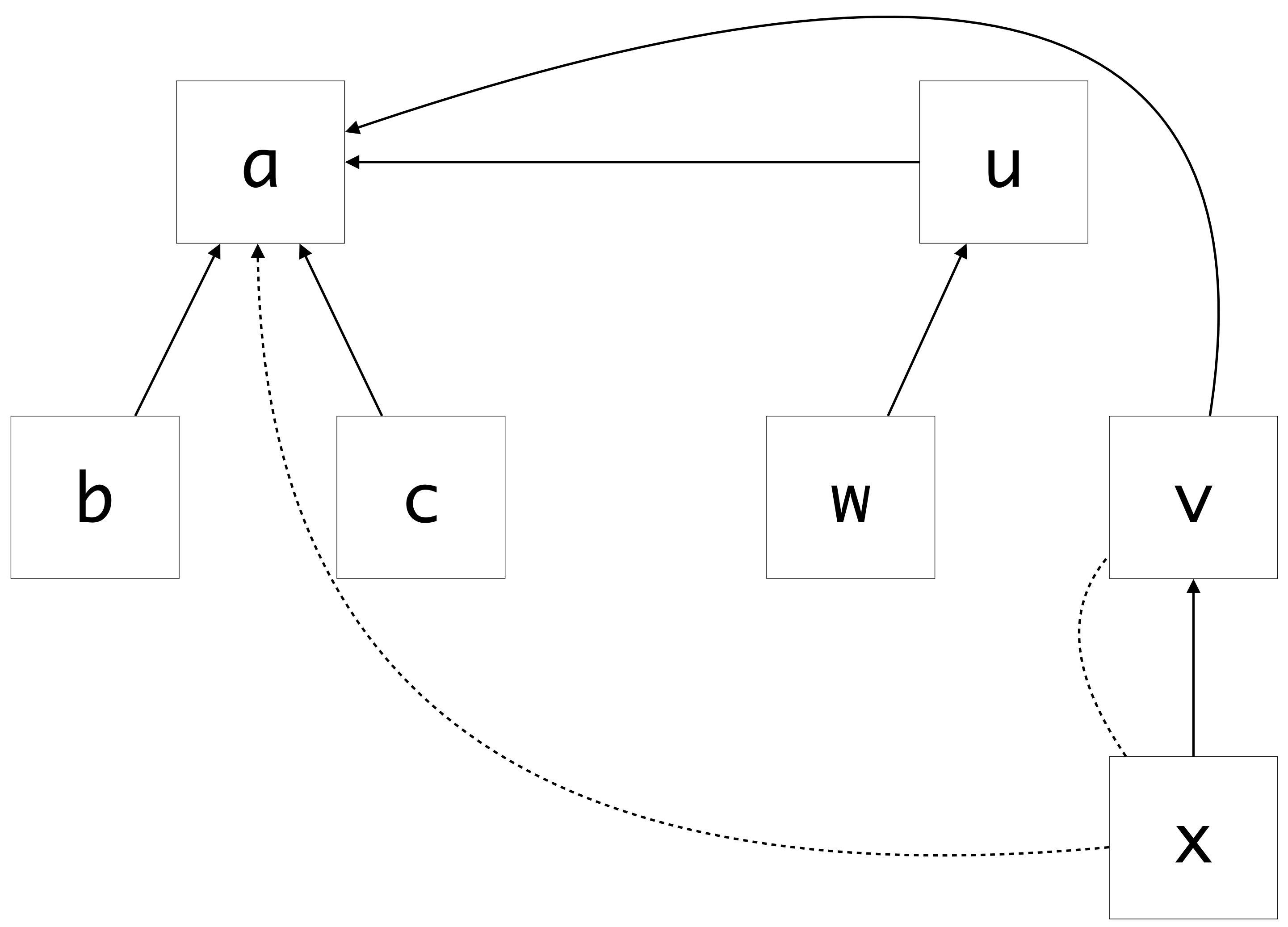

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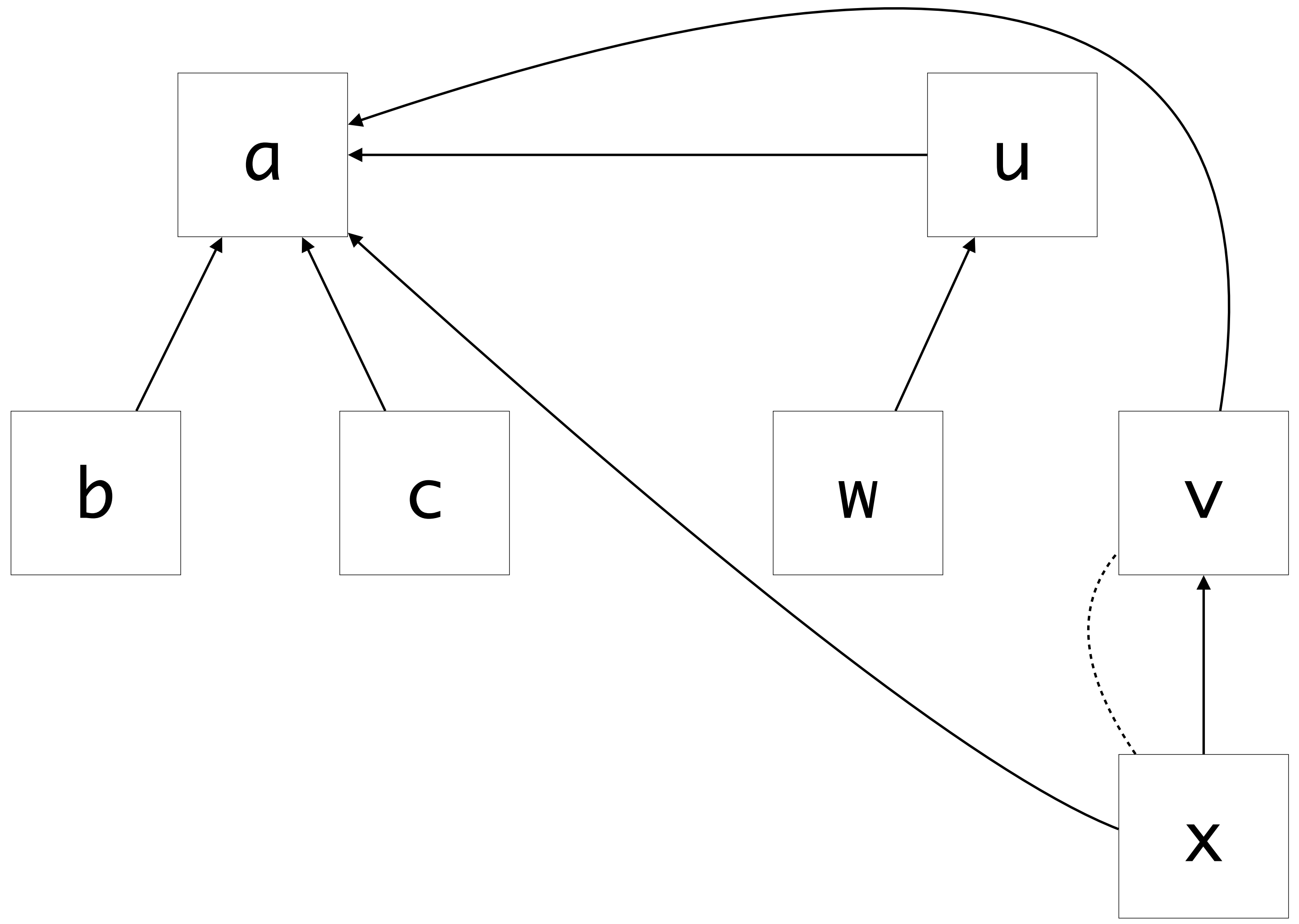

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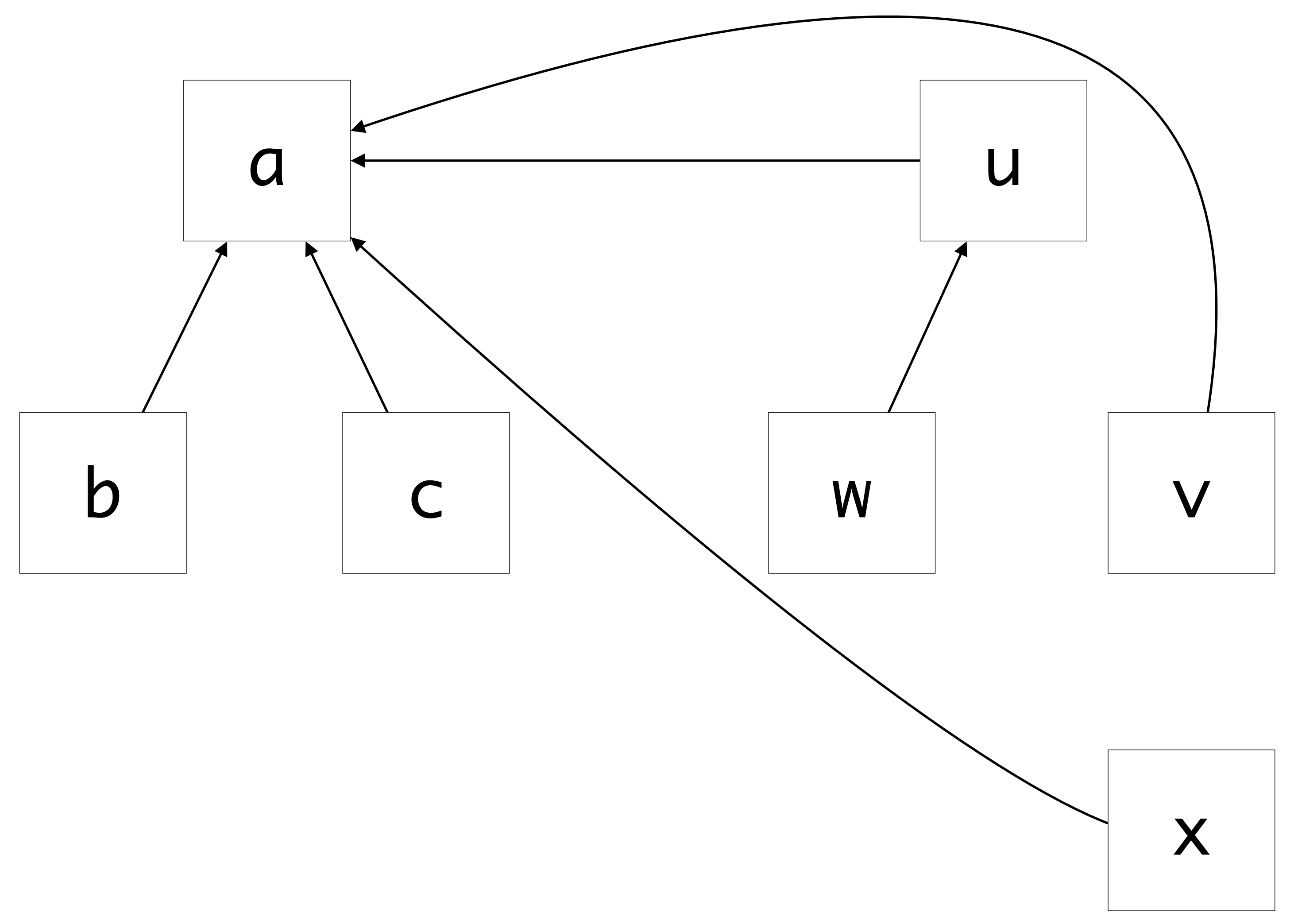

Tree Balancing

```
FIND(a):
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```
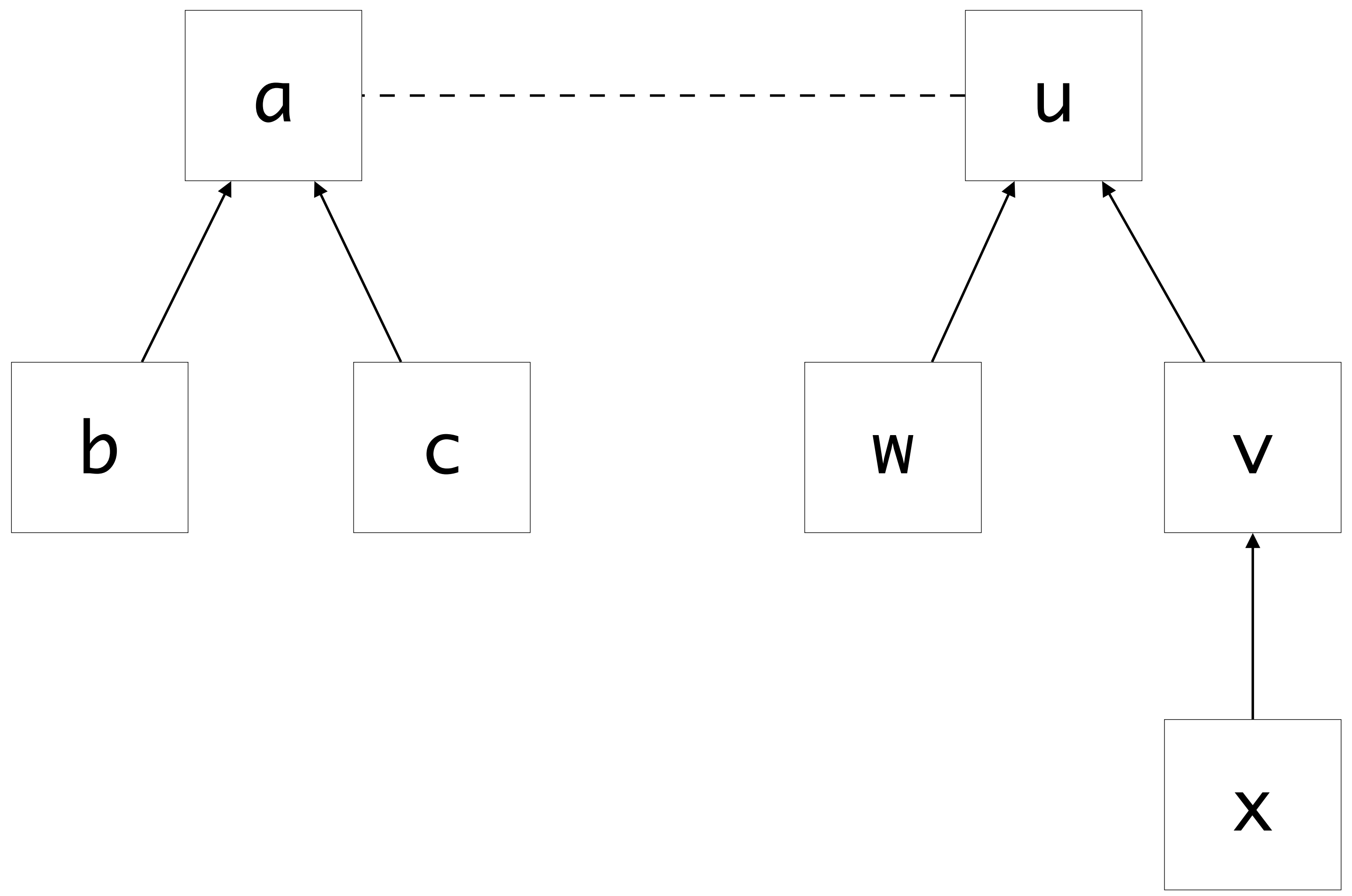

…

x == c



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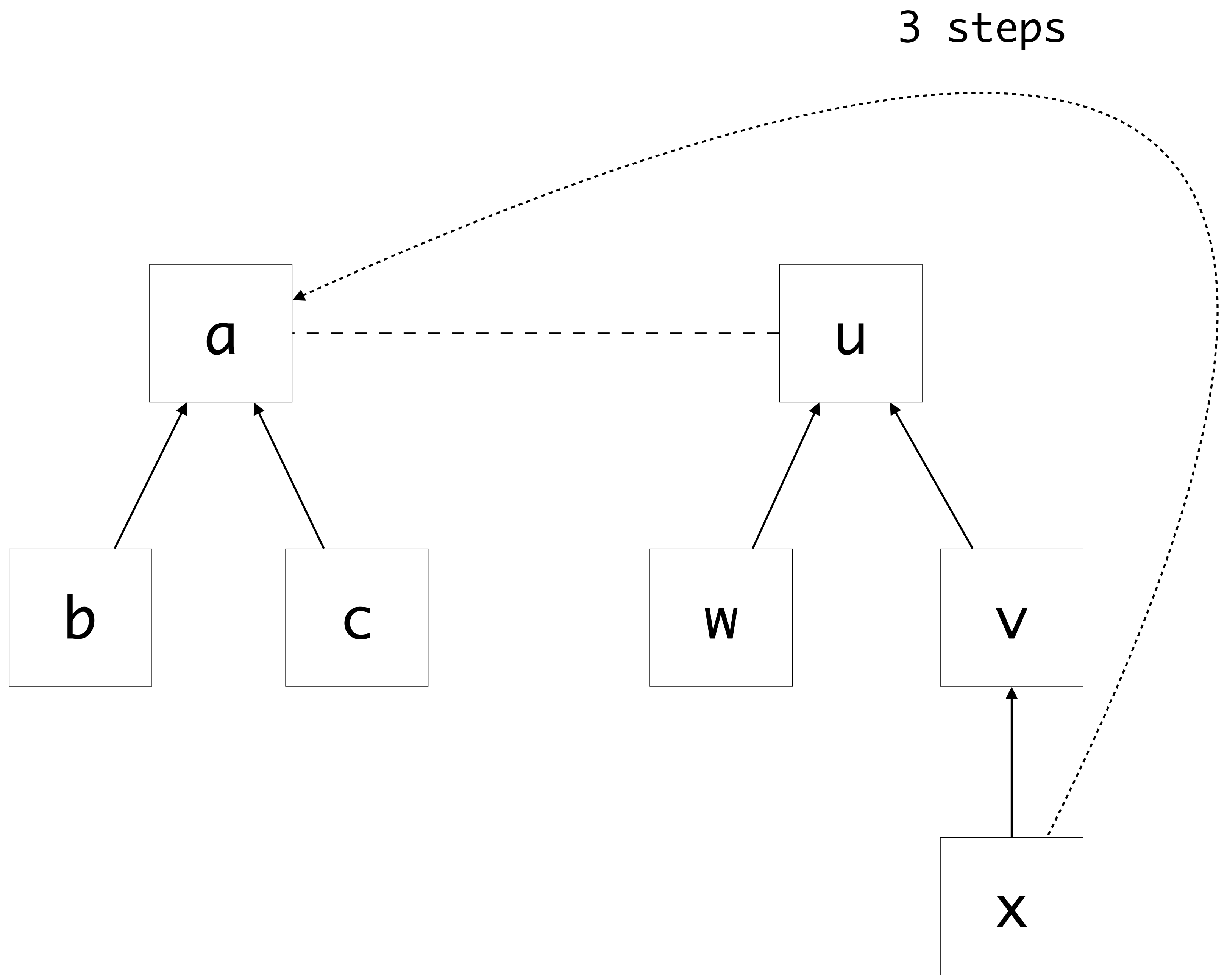

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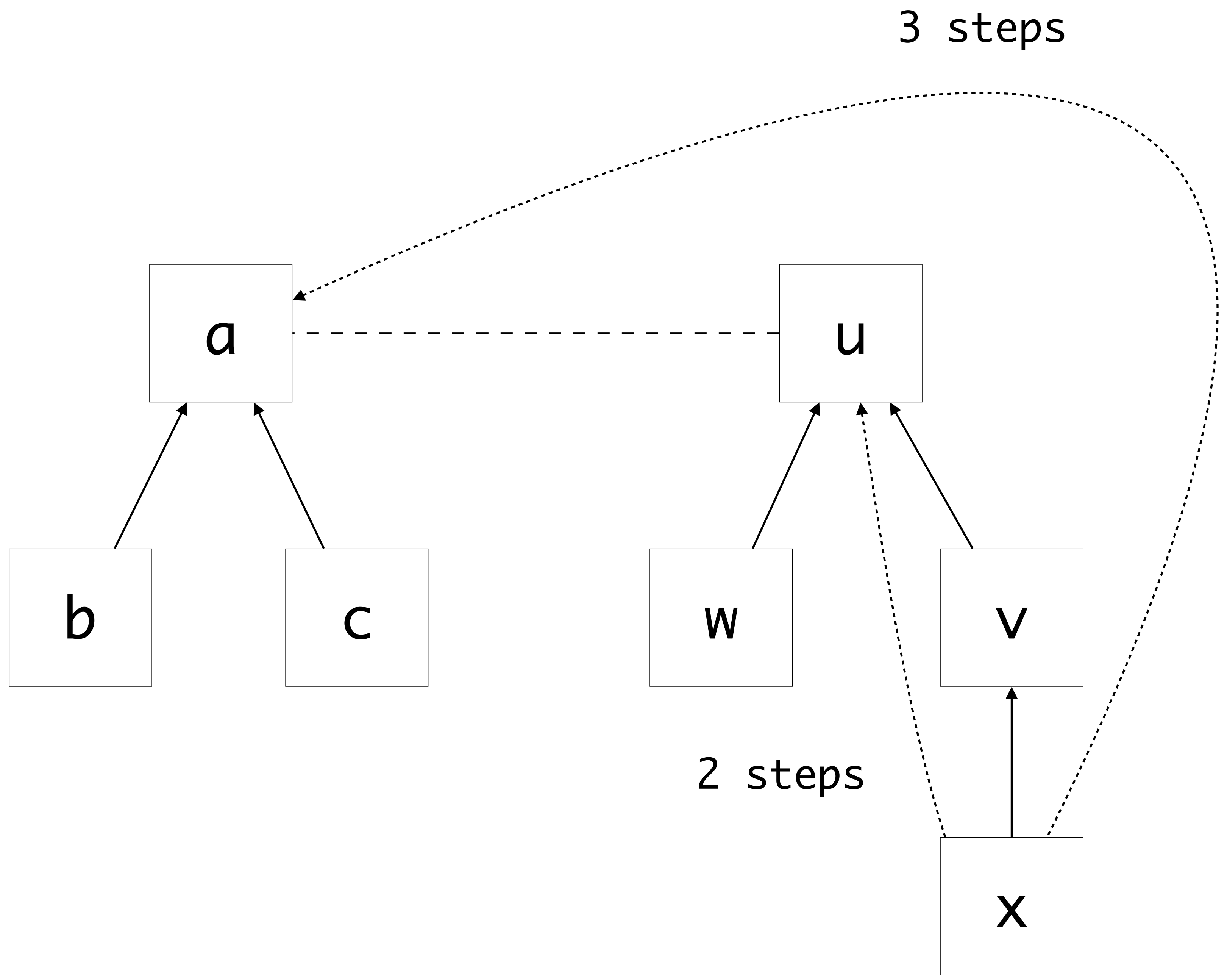

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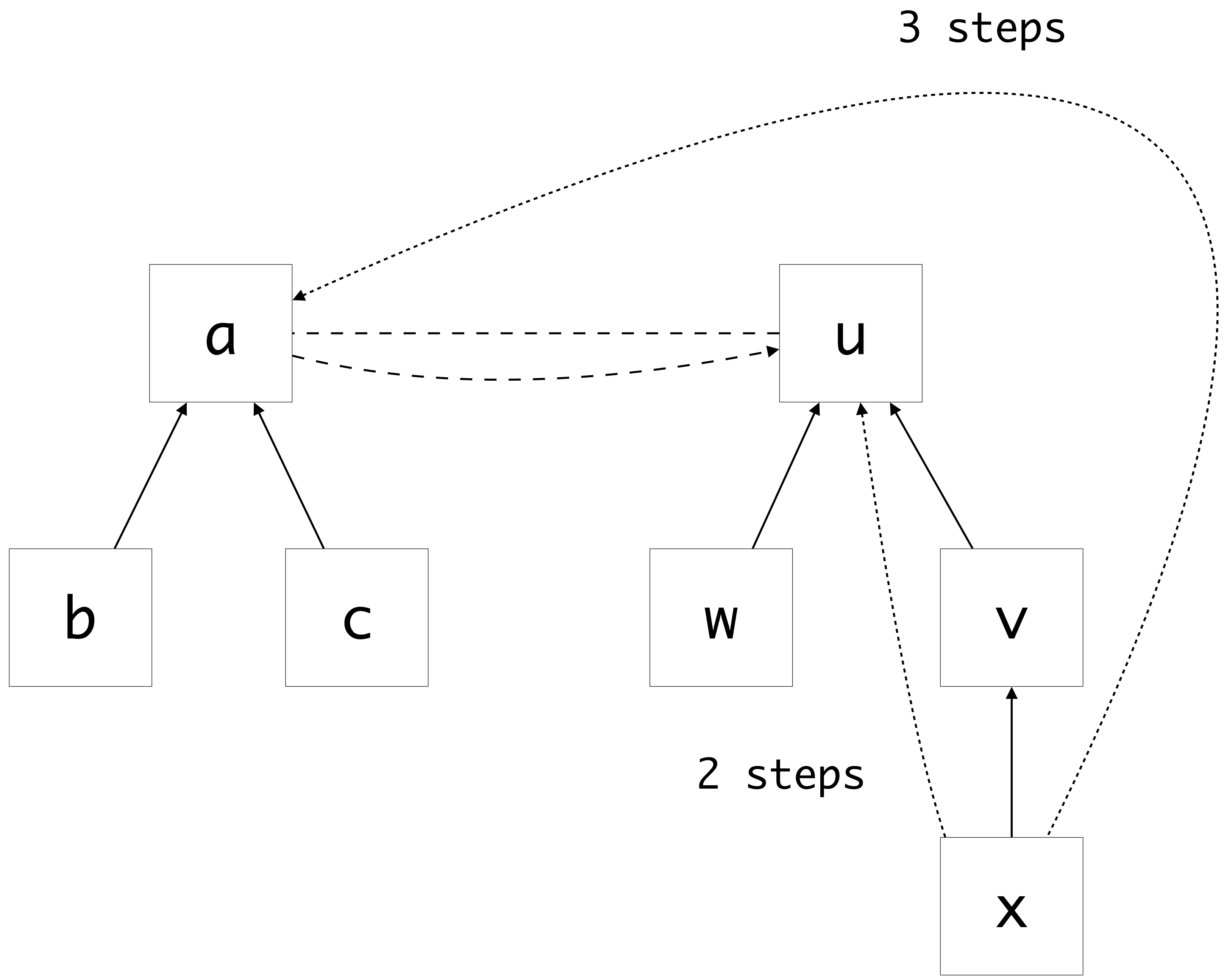

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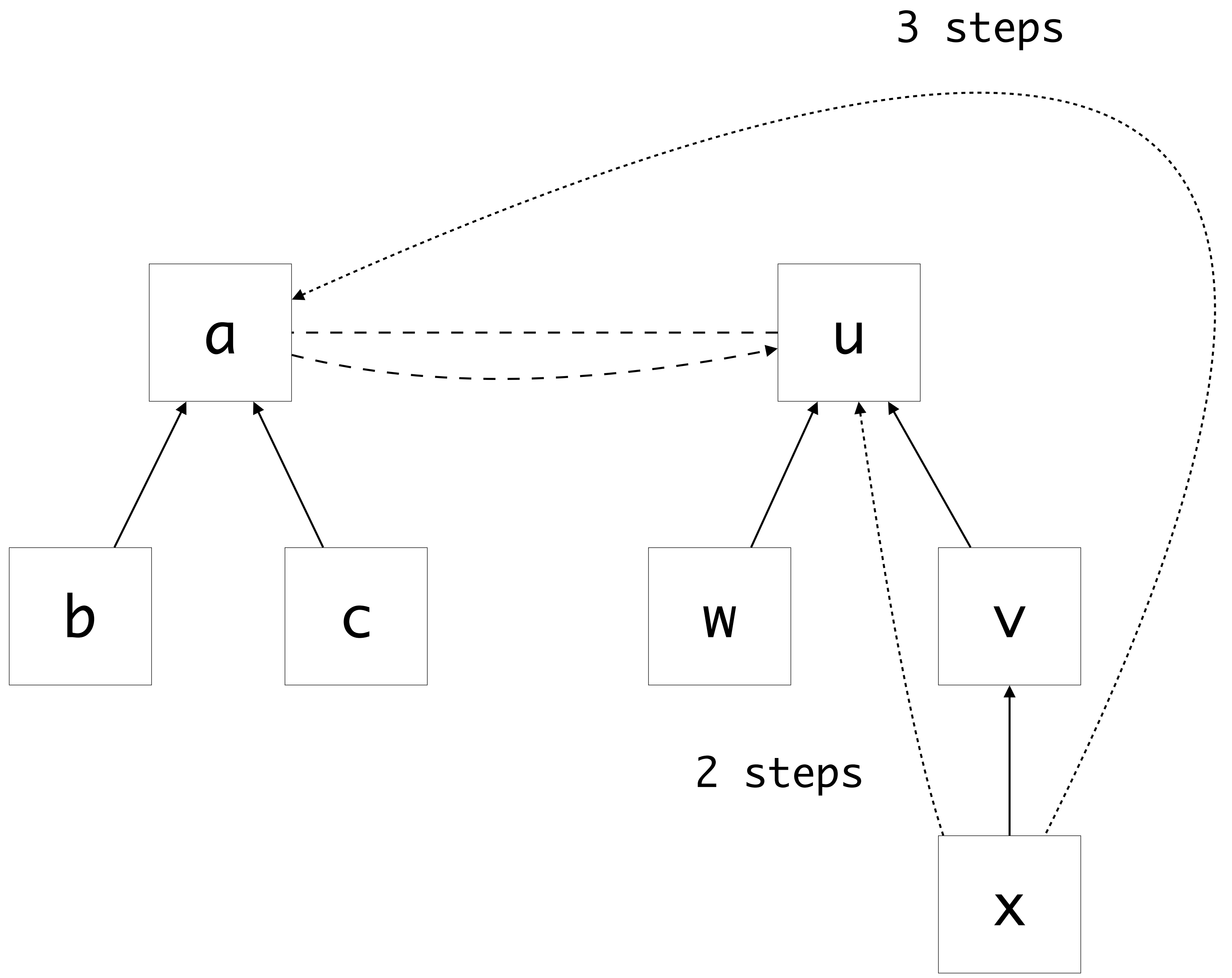

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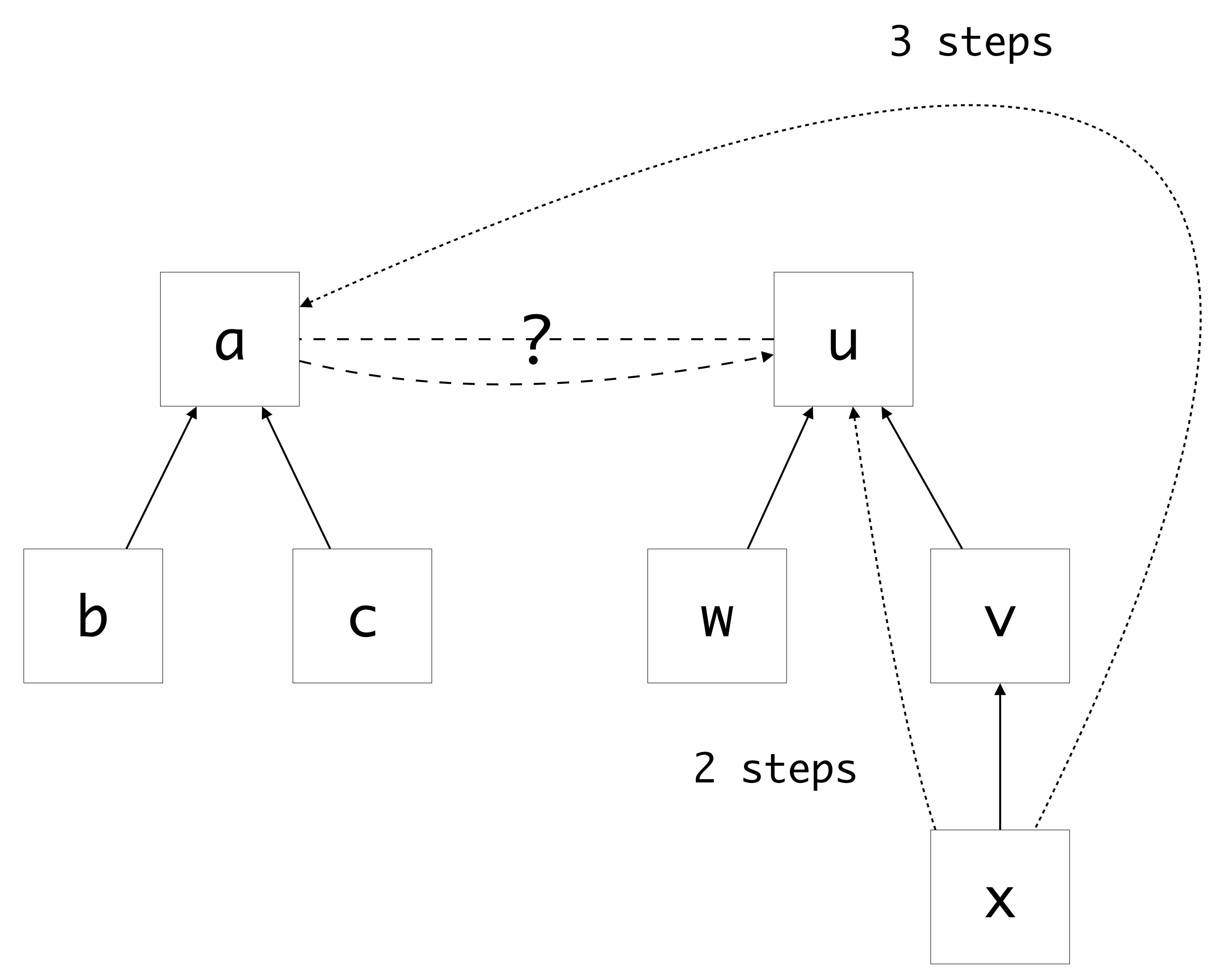

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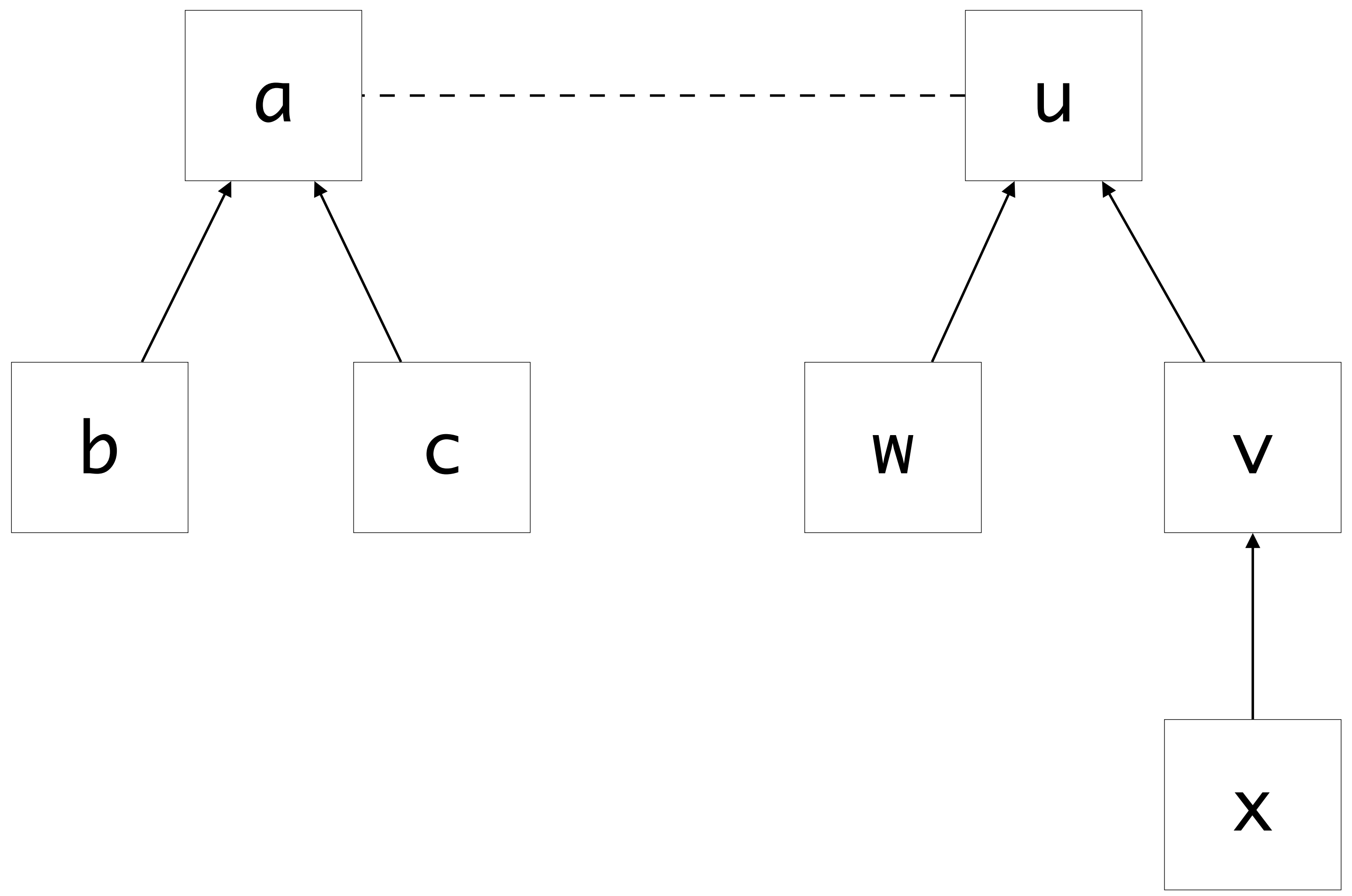

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  b1 := FIND(a1)
  b2 := FIND(a2)
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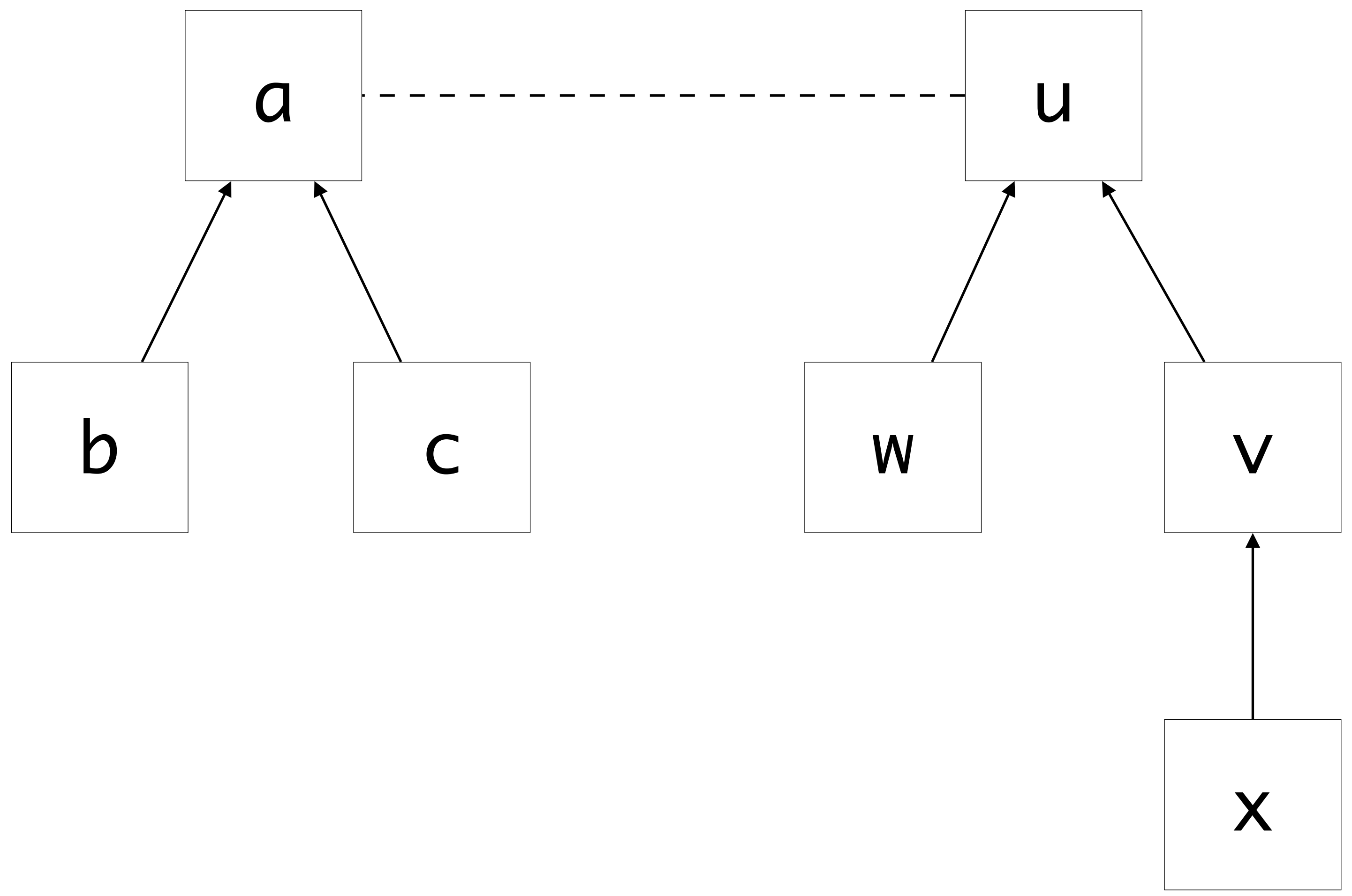
LINK(a1,a2):
  if size(a2) > size(a1):
     rep(a1) := a2
     size(a2) += size(a1)
  else:
     rep(a2) := a1
     size(a1) += size(a2)
```

…

x == c



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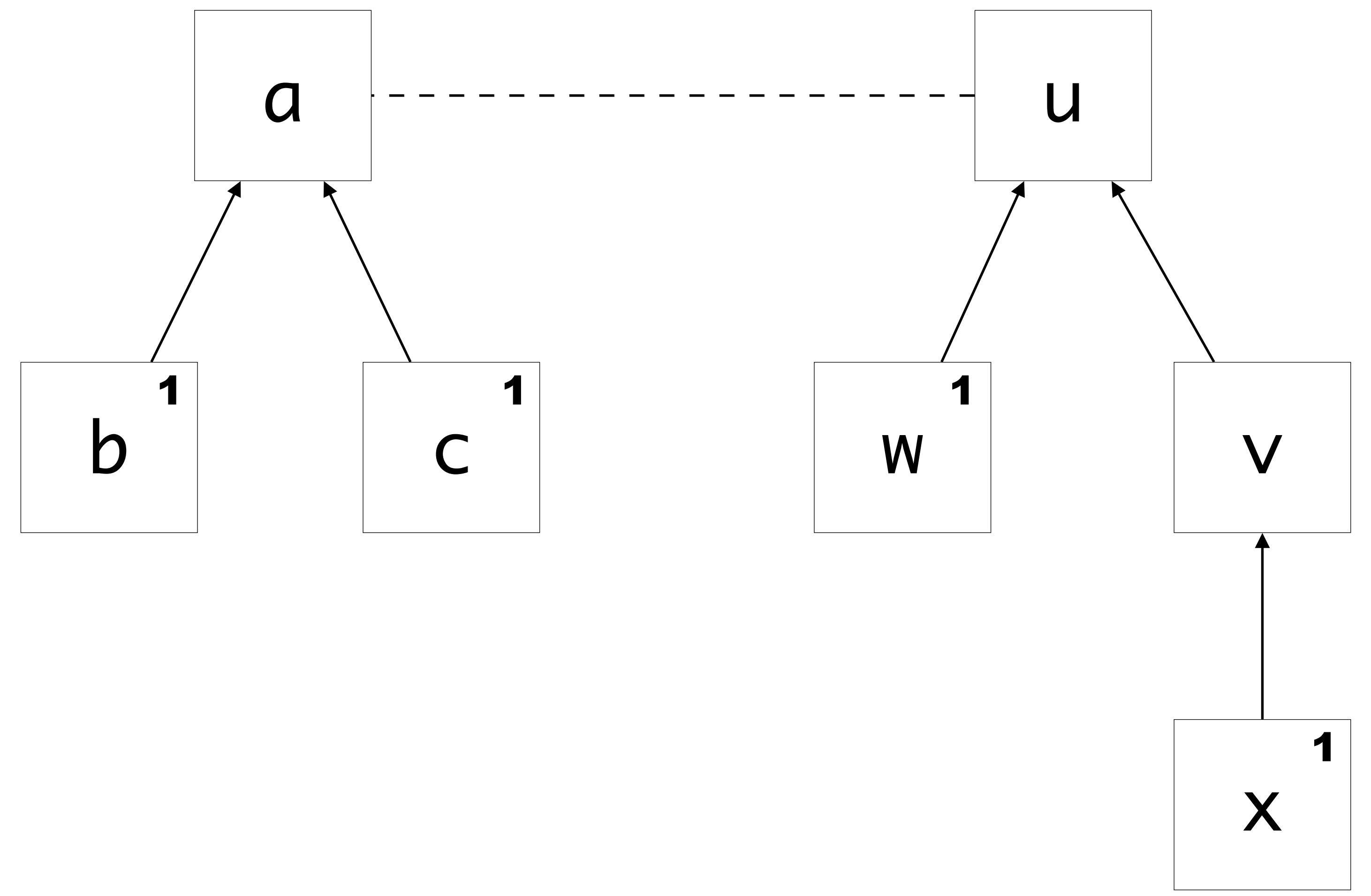

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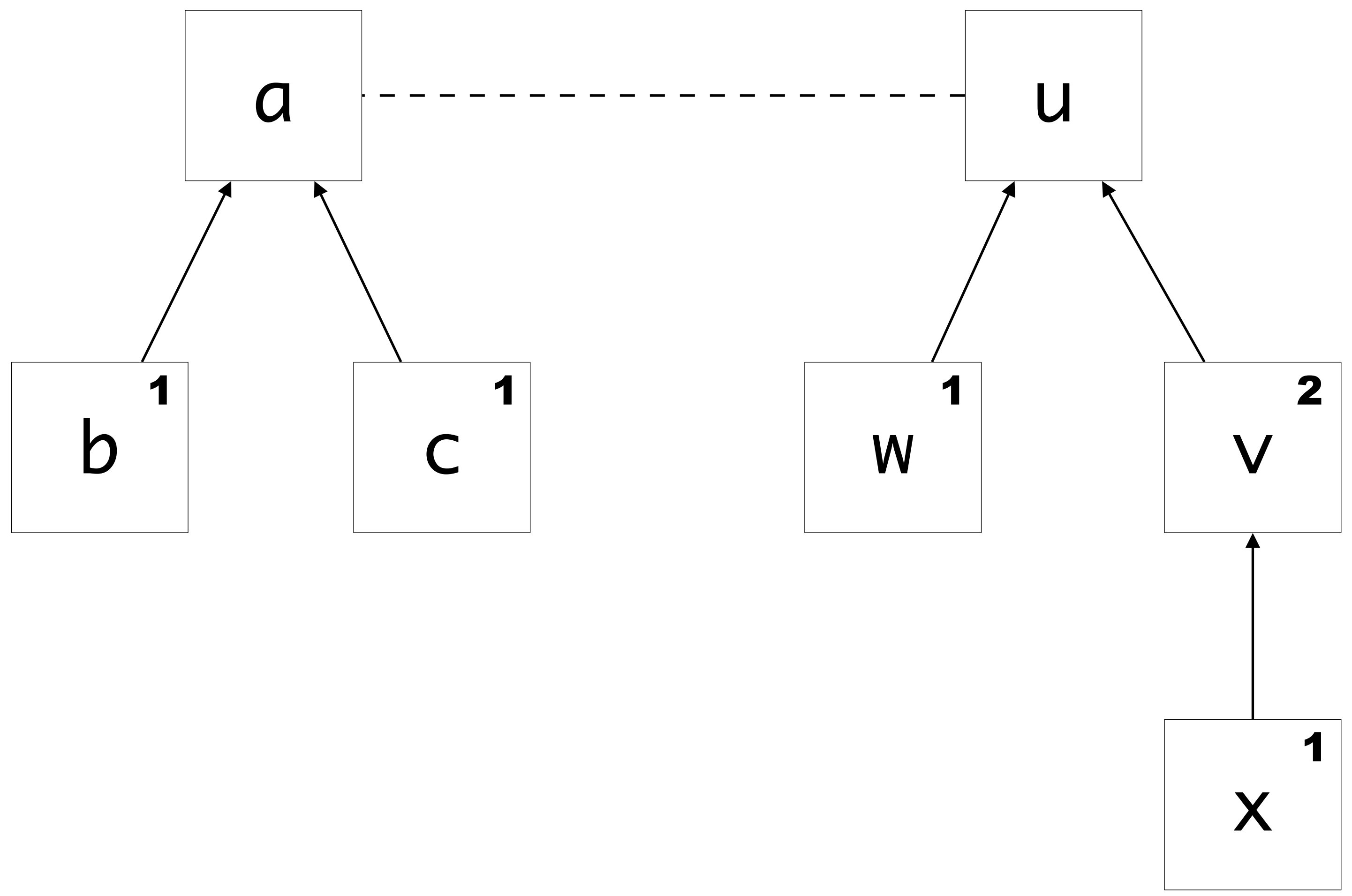
…

x == c



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```
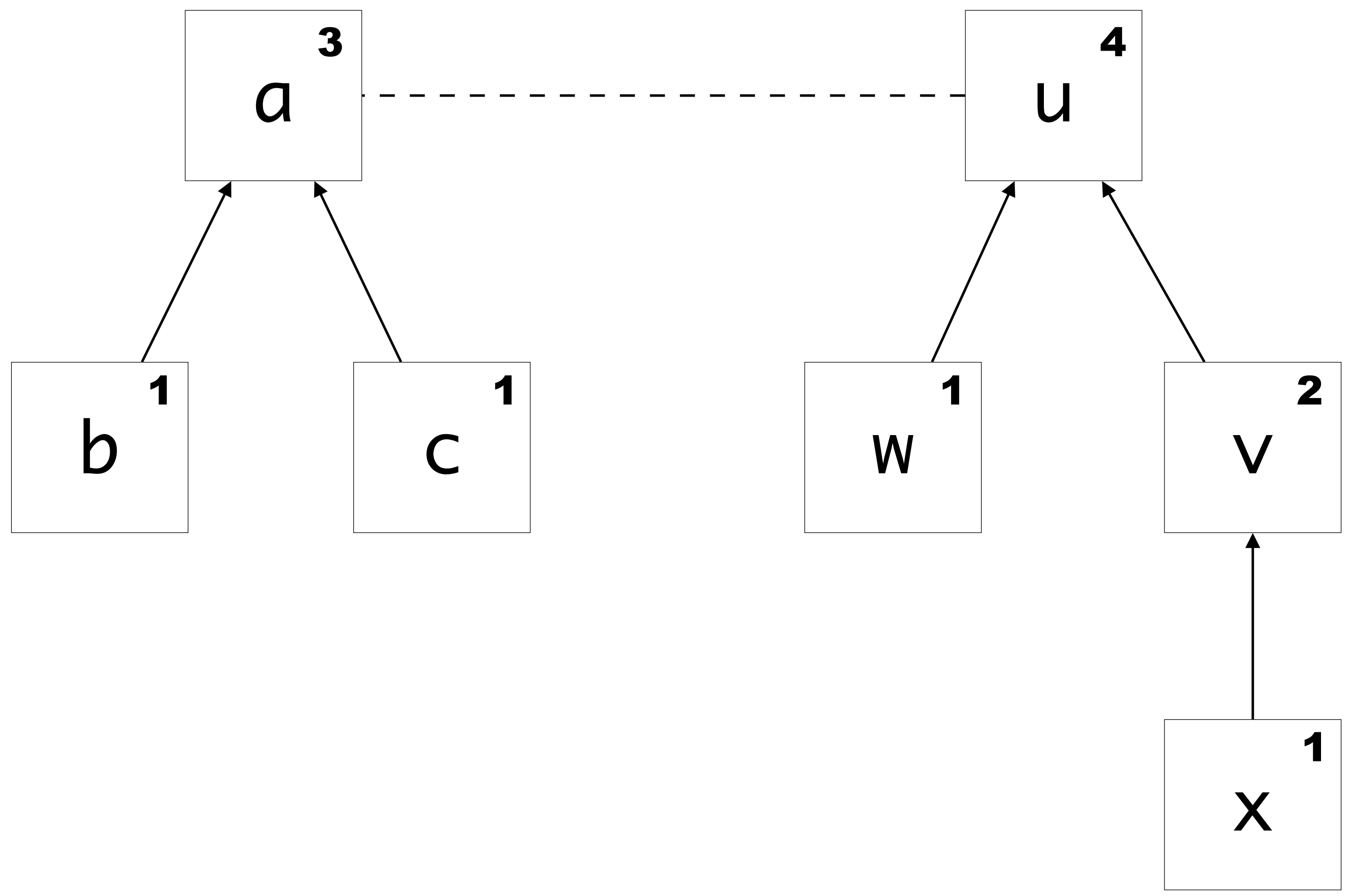

…

x == c



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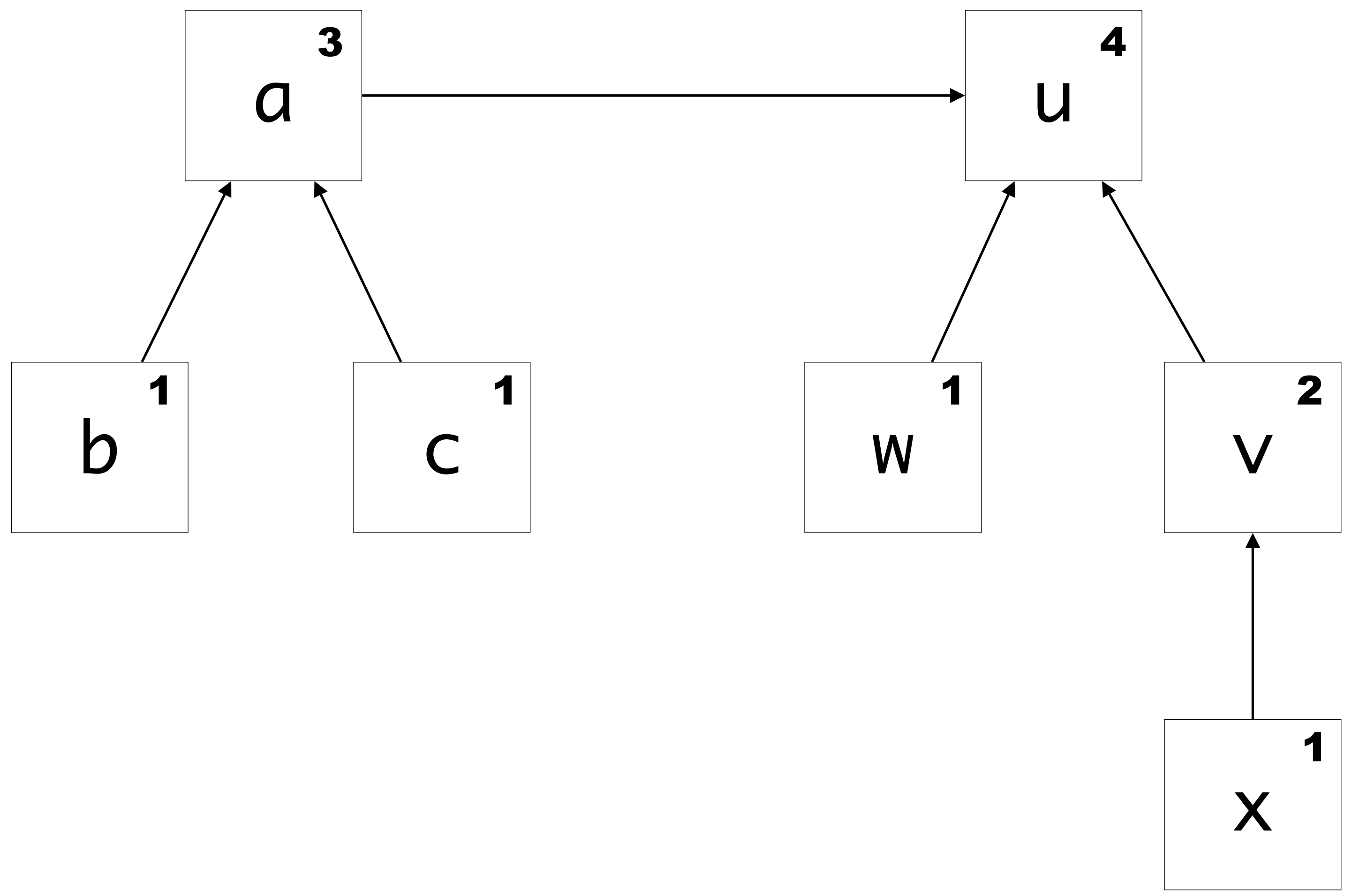
…

x == c



The Complex Case

```
h(a₁          , …,aₙ            , f(b₀,b₀), …, f(bₙ₋₁,bₙ₋₁), aₙ) ==
h(f(a₀,a₀), …,f(aₙ₋₁,aₙ₋₁), b₁,          …, bₙ₋₁          , bₙ)
```

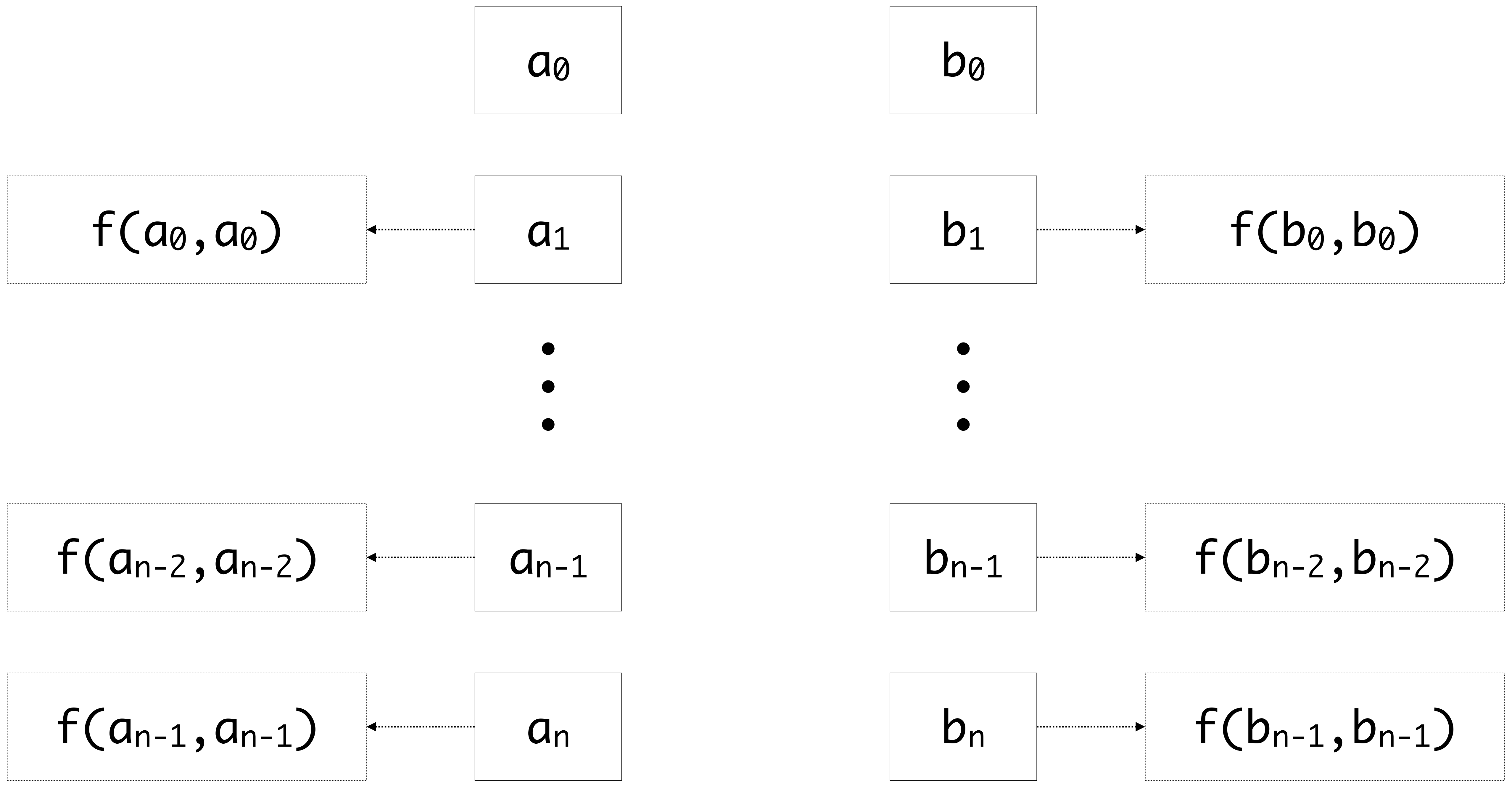
The Complex Case

$$h(a_1, \dots, a_n, f(b_0,b_0), \dots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \dots, f(a_{n-1},a_{n-1}), b_1, \dots, b_{n-1}, b_n)$$

	a_0	b_0	
$f(a_0,a_0)$	a_1	b_1	$f(b_0,b_0)$
	$\vdots$	$\vdots$	
$f(a_{n-2},a_{n-2})$	a_{n-1}	b_{n-1}	$f(b_{n-2},b_{n-2})$
$f(a_{n-1},a_{n-1})$	a_n	b_n	$f(b_{n-1},b_{n-1})$

The Complex Case

$$
\begin{aligned}
&h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) == \\
&h(f(a_0,a_0), \ldots, f(a_{n-1},a_{n-1}), b_1, \ldots, b_{n-1}, b_n)
\end{aligned}
$$



The Complex Case

$$h(a_1, \dots, a_n, f(b_0,b_0), \dots, f(b_{n-1},b_{n-1}), a_n) ==$$
$$h(f(a_0,a_0), \dots, f(a_{n-1},a_{n-1}), b_1, \dots, b_{n-1}, b_n)$$

$$\begin{tikzcd}
 & a_0 & b_0 & \\
f(a_0,a_0) & a_1 \arrow[l] & b_1 \arrow[r] & f(b_0,b_0) \\
 & \vdots & \vdots & \\
f(a_{n-2},a_{n-2}) & a_{n-1} \arrow[l] & b_{n-1} \arrow[r] & f(b_{n-2},b_{n-2}) \\
f(a_{n-1},a_{n-1}) & a_n \arrow[l] & b_n \arrow[r] & f(b_{n-1},b_{n-1})
\end{tikzcd}$$

$a_n == b_n$

The Complex Case

$$h(a_1, \ldots, a_n, f(b_0,b_0), \ldots, f(b_{n-1},b_{n-1}), a_n) ==$$
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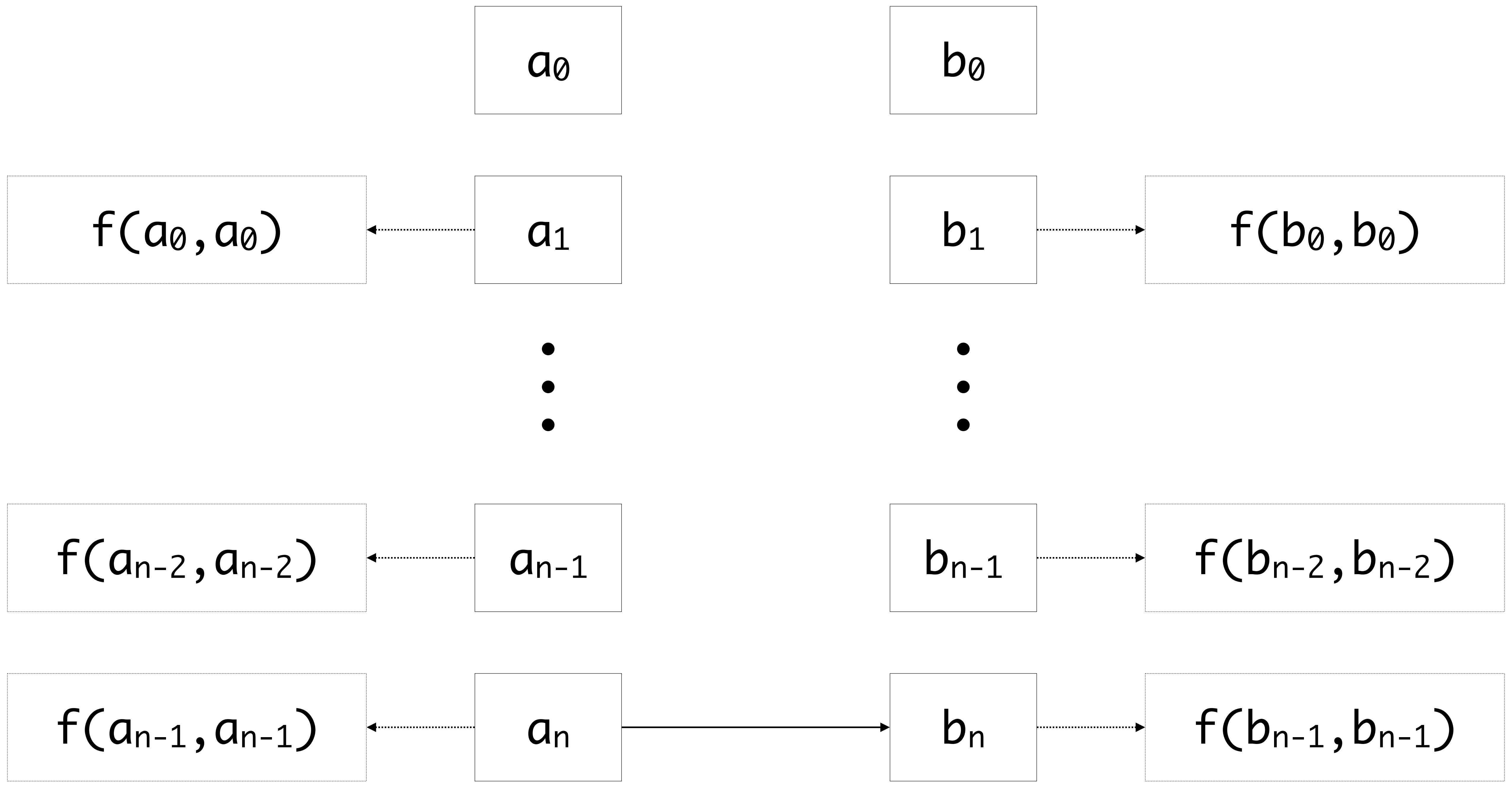

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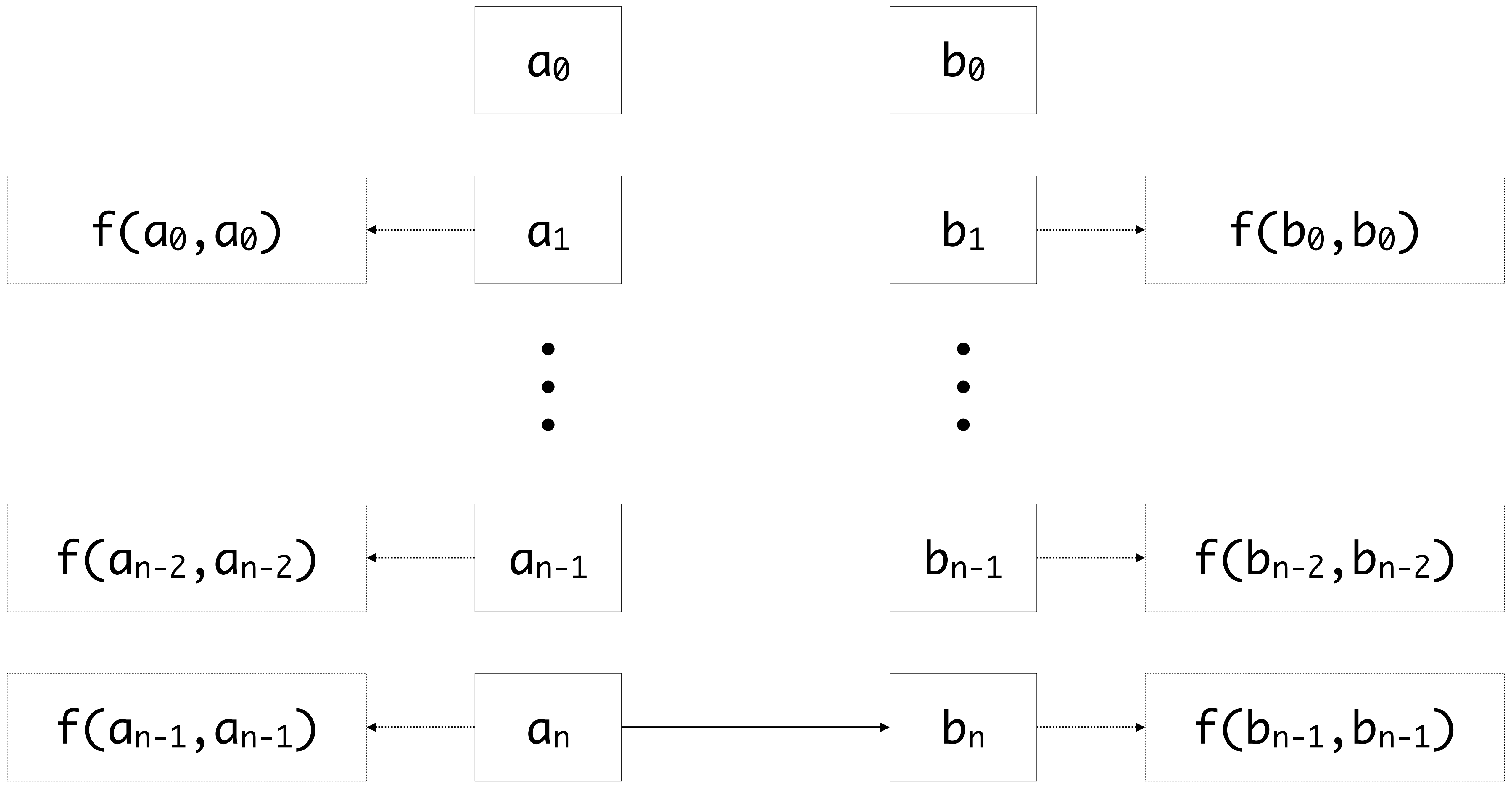

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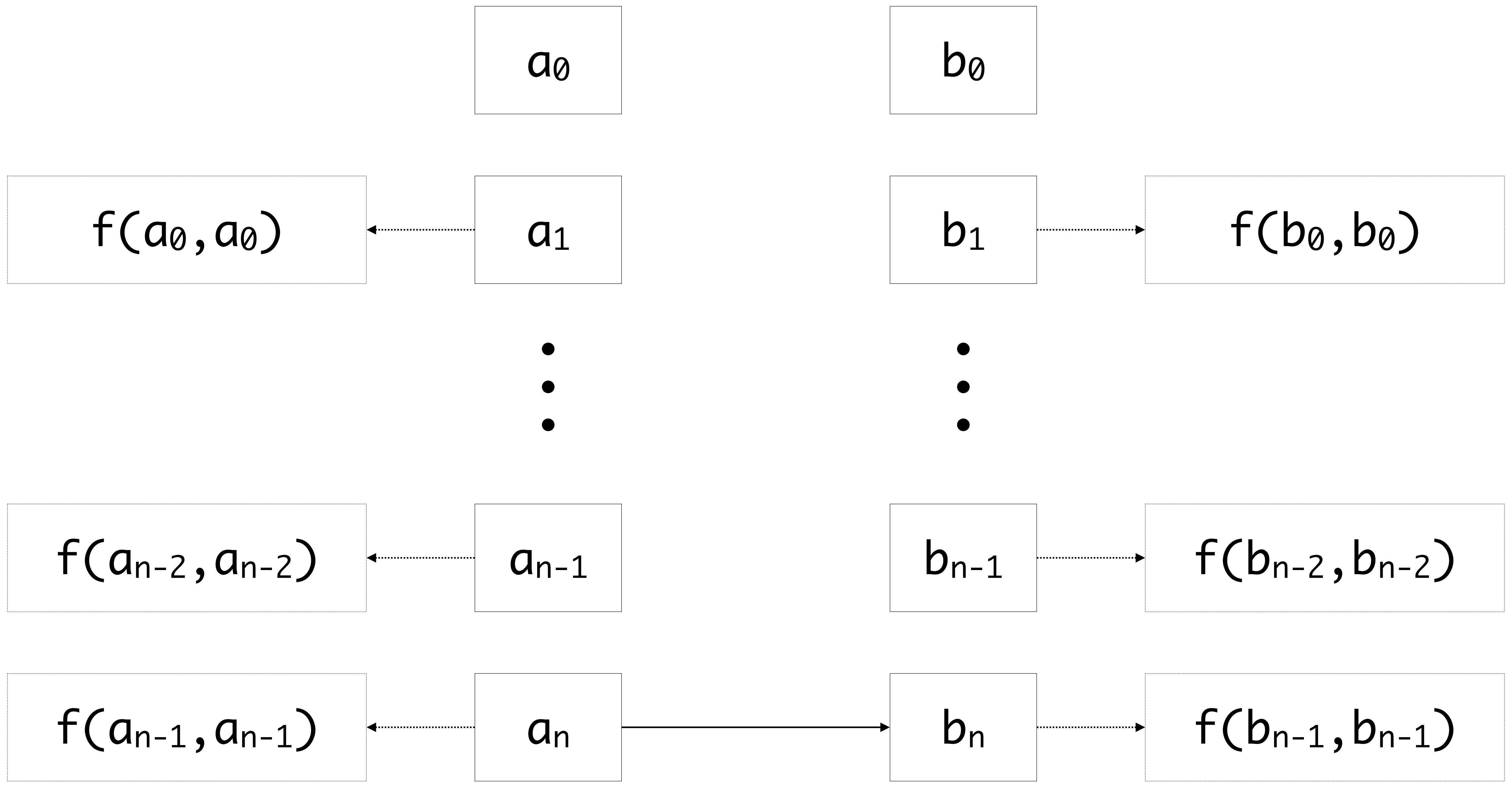

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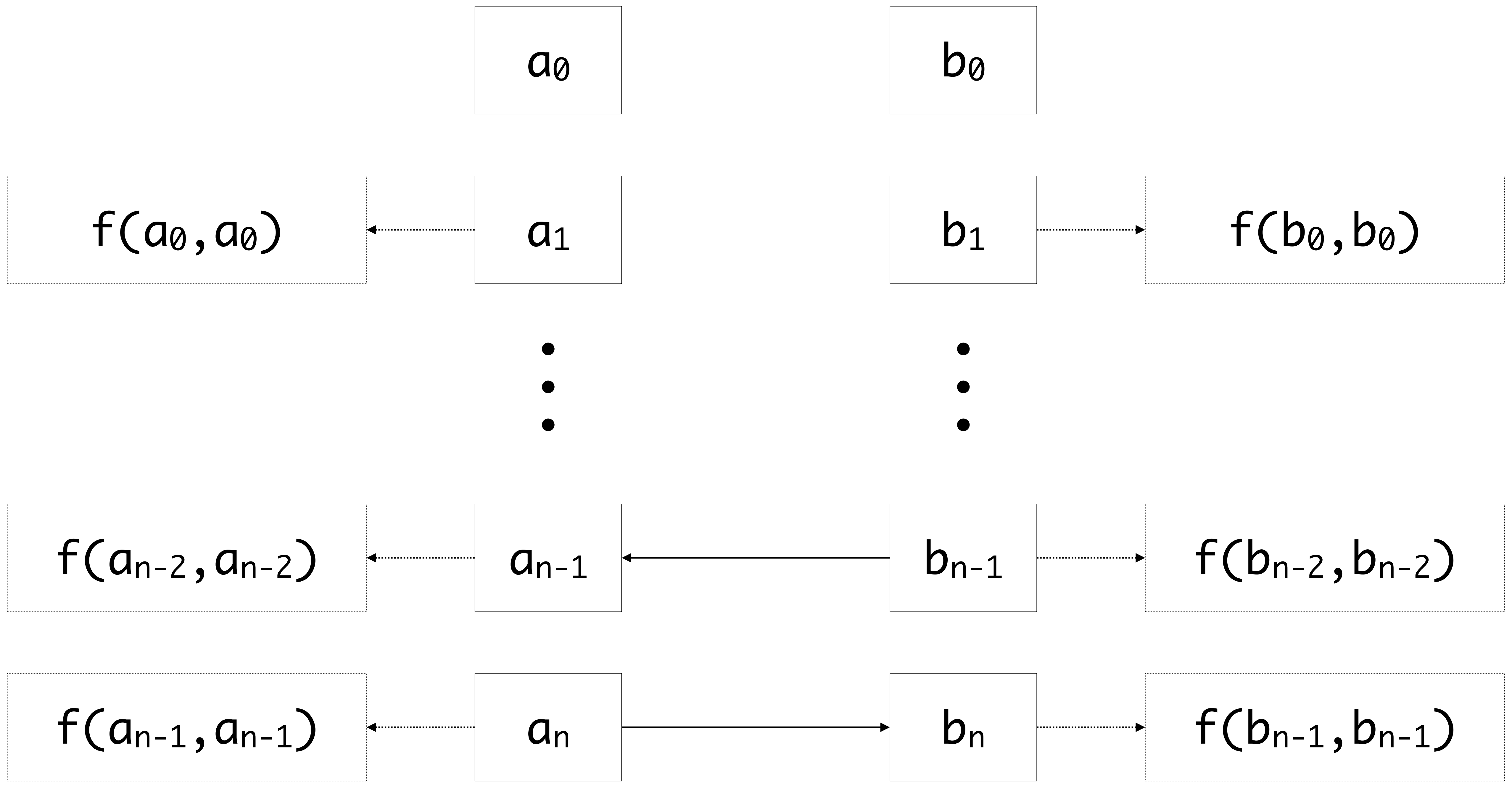

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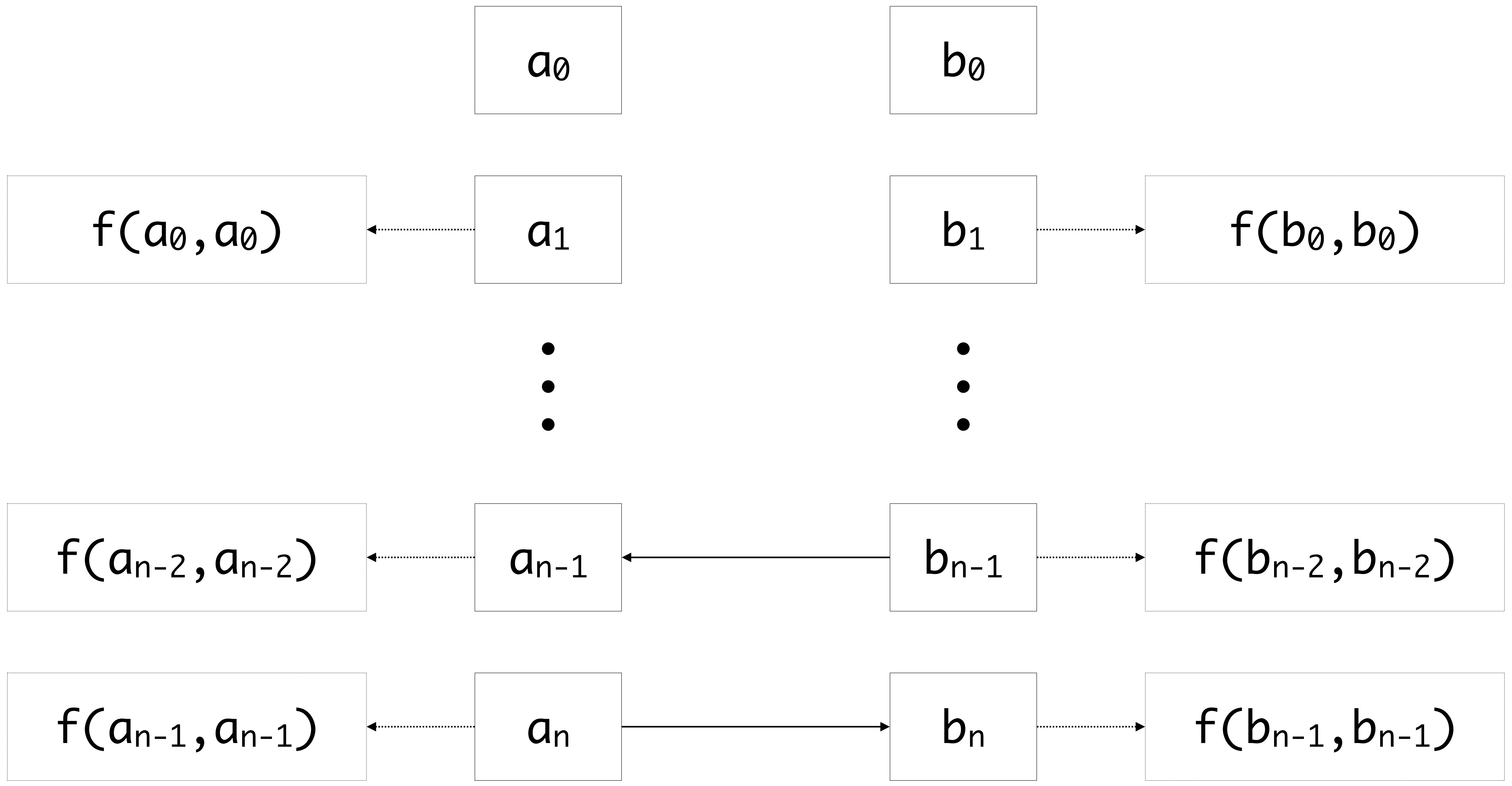

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$$
\begin{tikzcd}
 & a_0 & b_0 & \\
f(a_0,a_0) & a_1 \arrow[l, dotted] & b_1 \arrow[r, dotted] & f(b_0,b_0) \\
 & \vdots & \vdots & \\
f(a_{n-2},a_{n-2}) & a_{n-1} \arrow[l, dotted] & b_{n-1} \arrow[l] \arrow[r, dotted] & f(b_{n-2},b_{n-2}) \\
f(a_{n-1},a_{n-1}) & a_n \arrow[l, dotted] \arrow[r] & b_n \arrow[r, dotted] & f(b_{n-1},b_{n-1})
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$$

$$
\begin{gathered}
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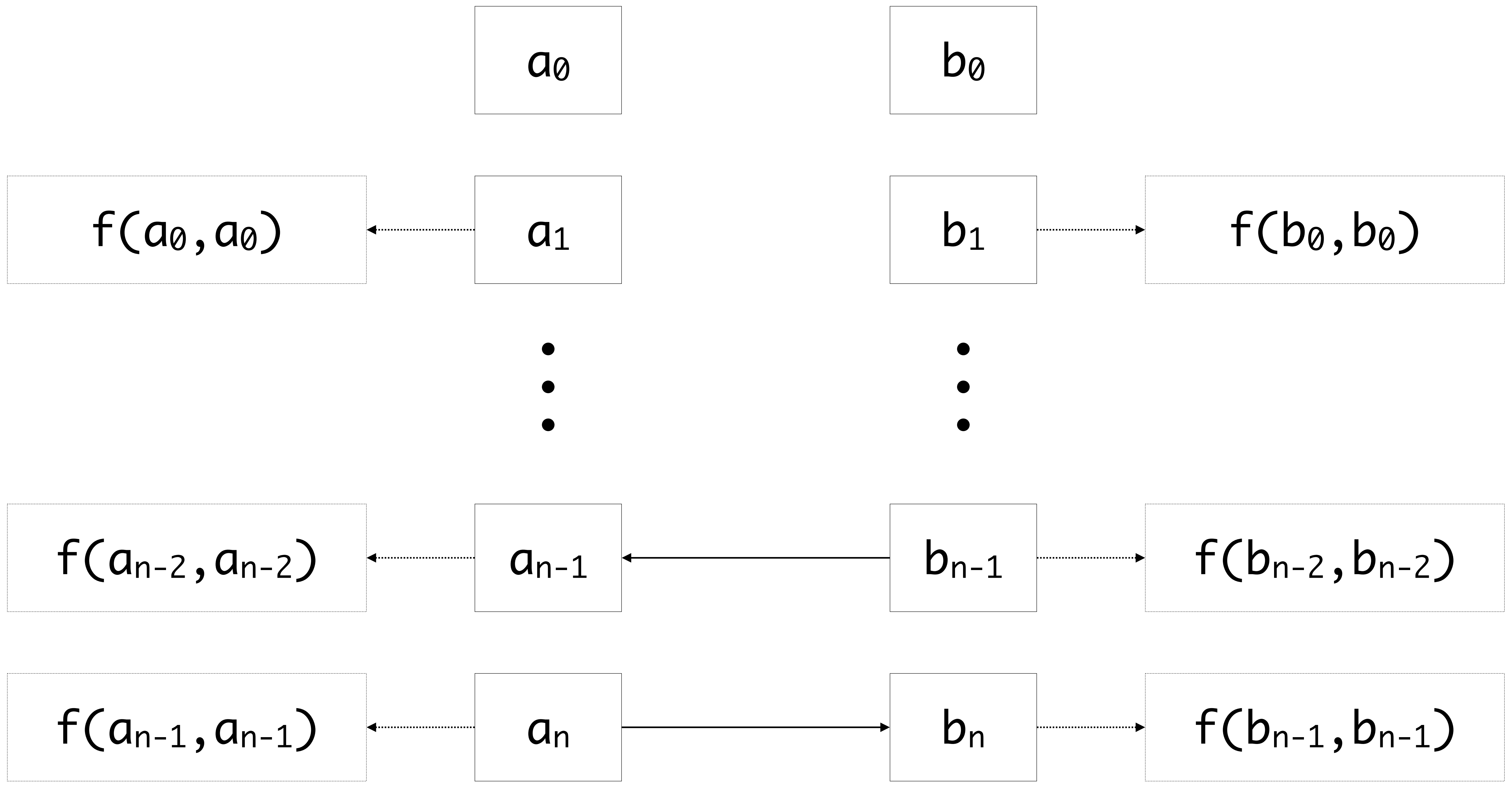

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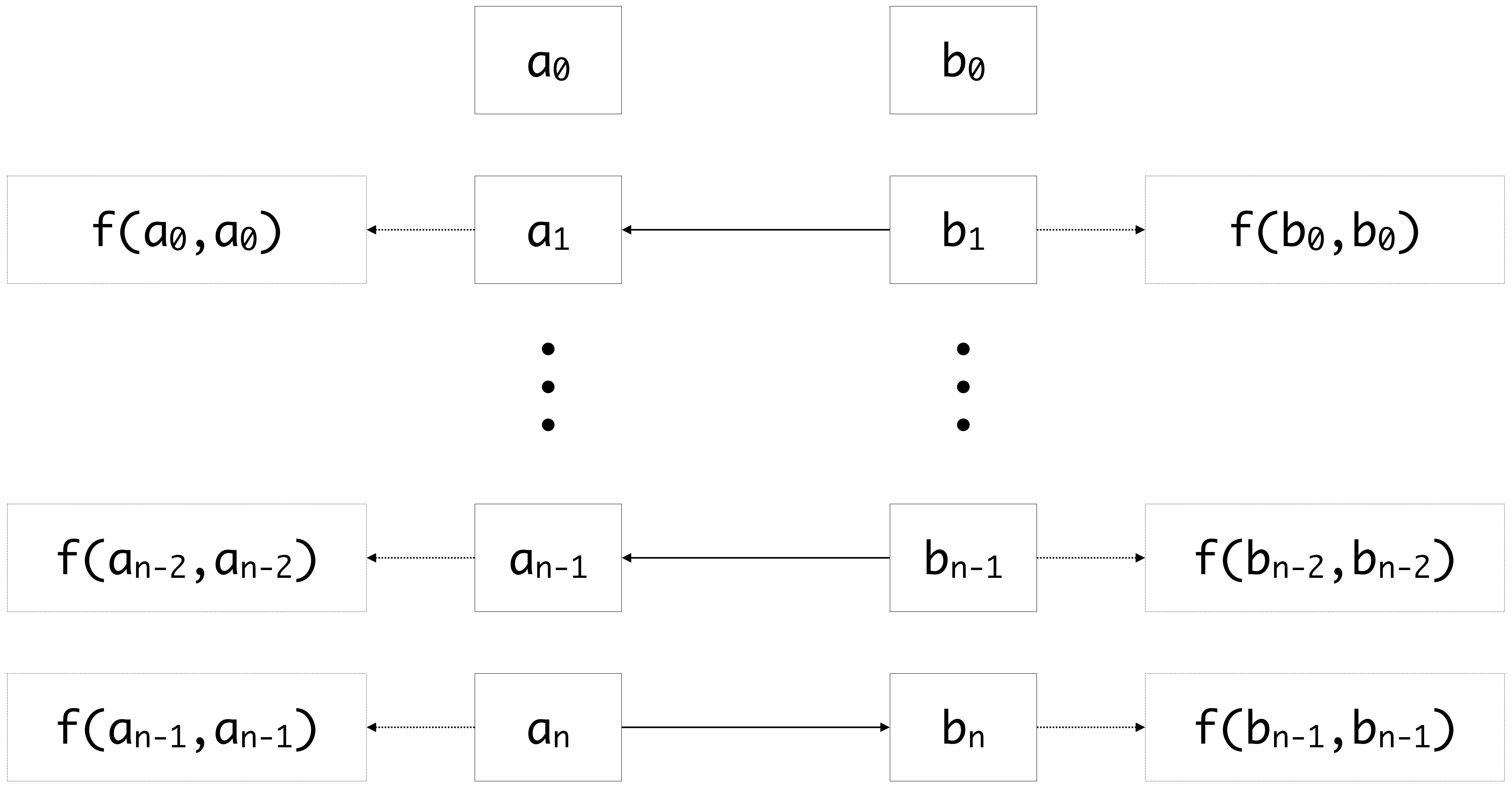

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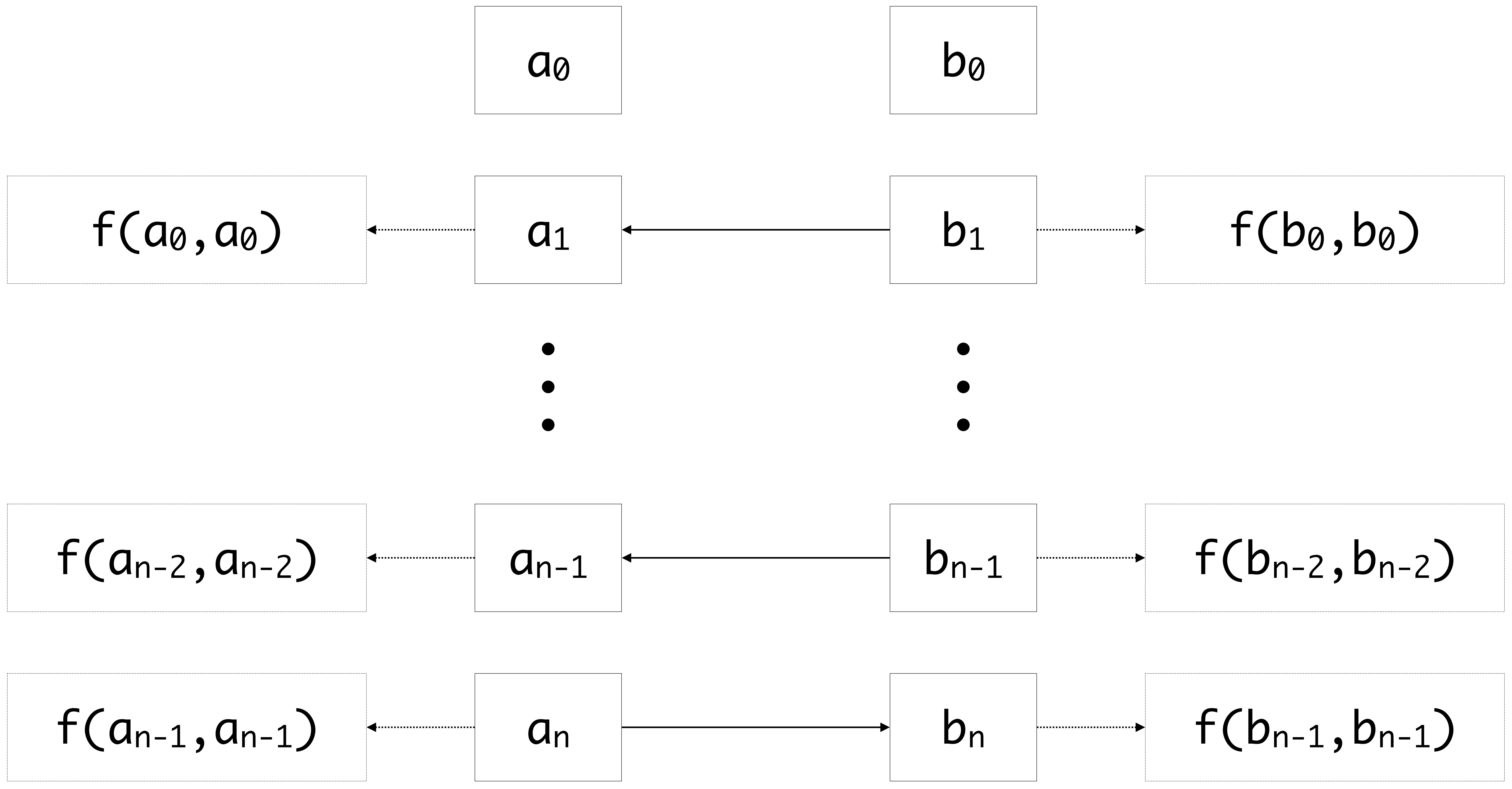

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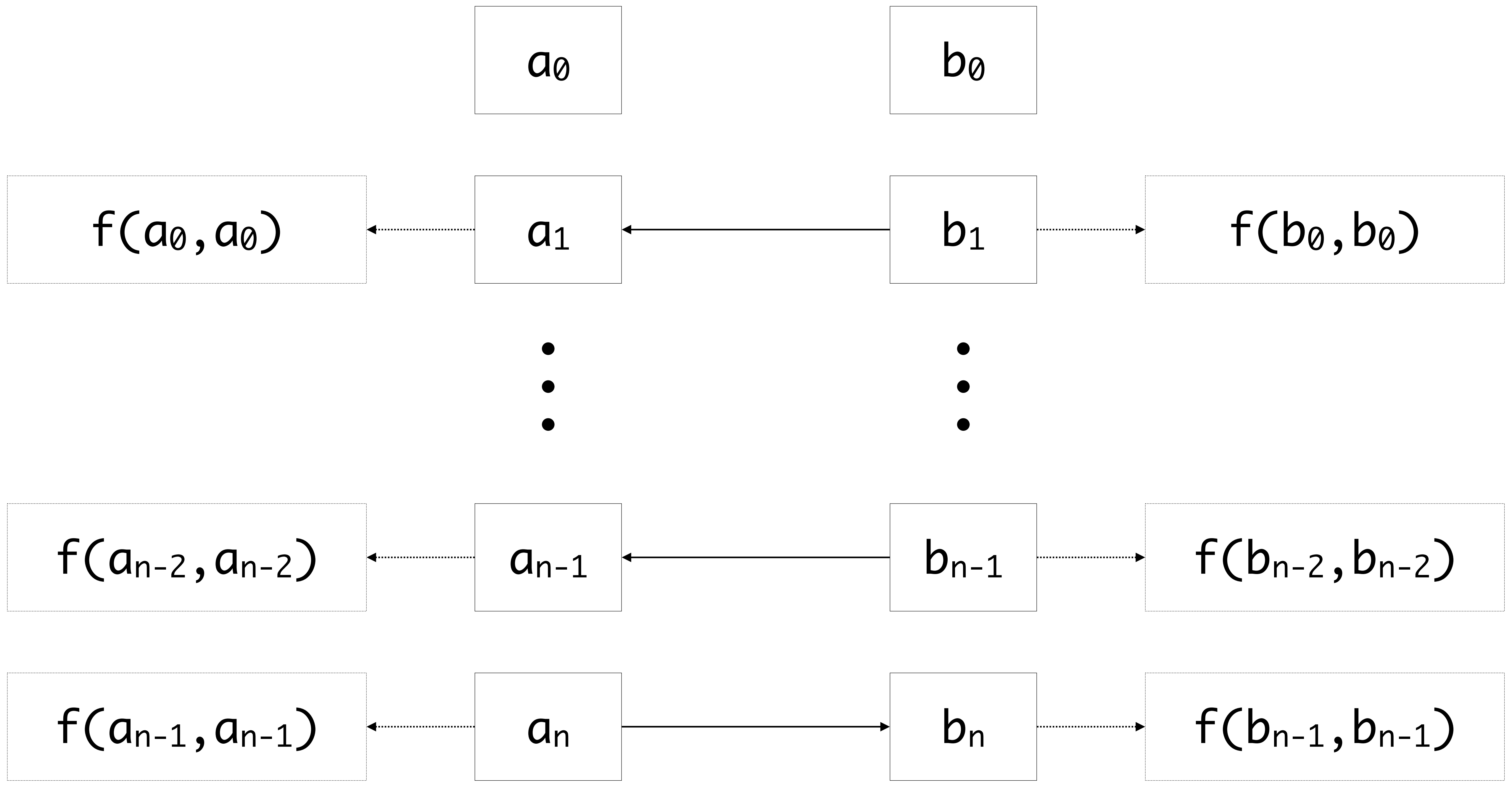

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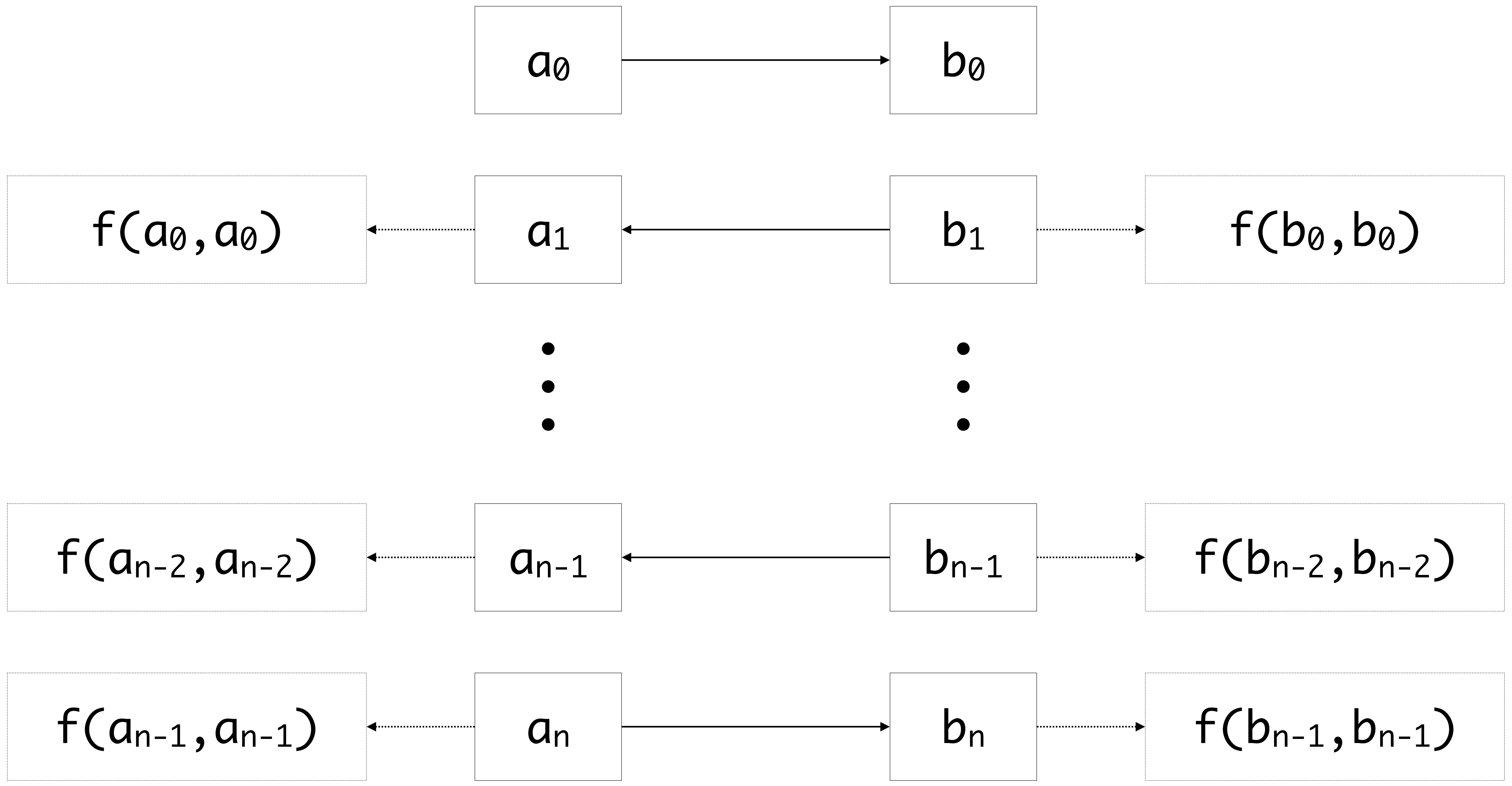

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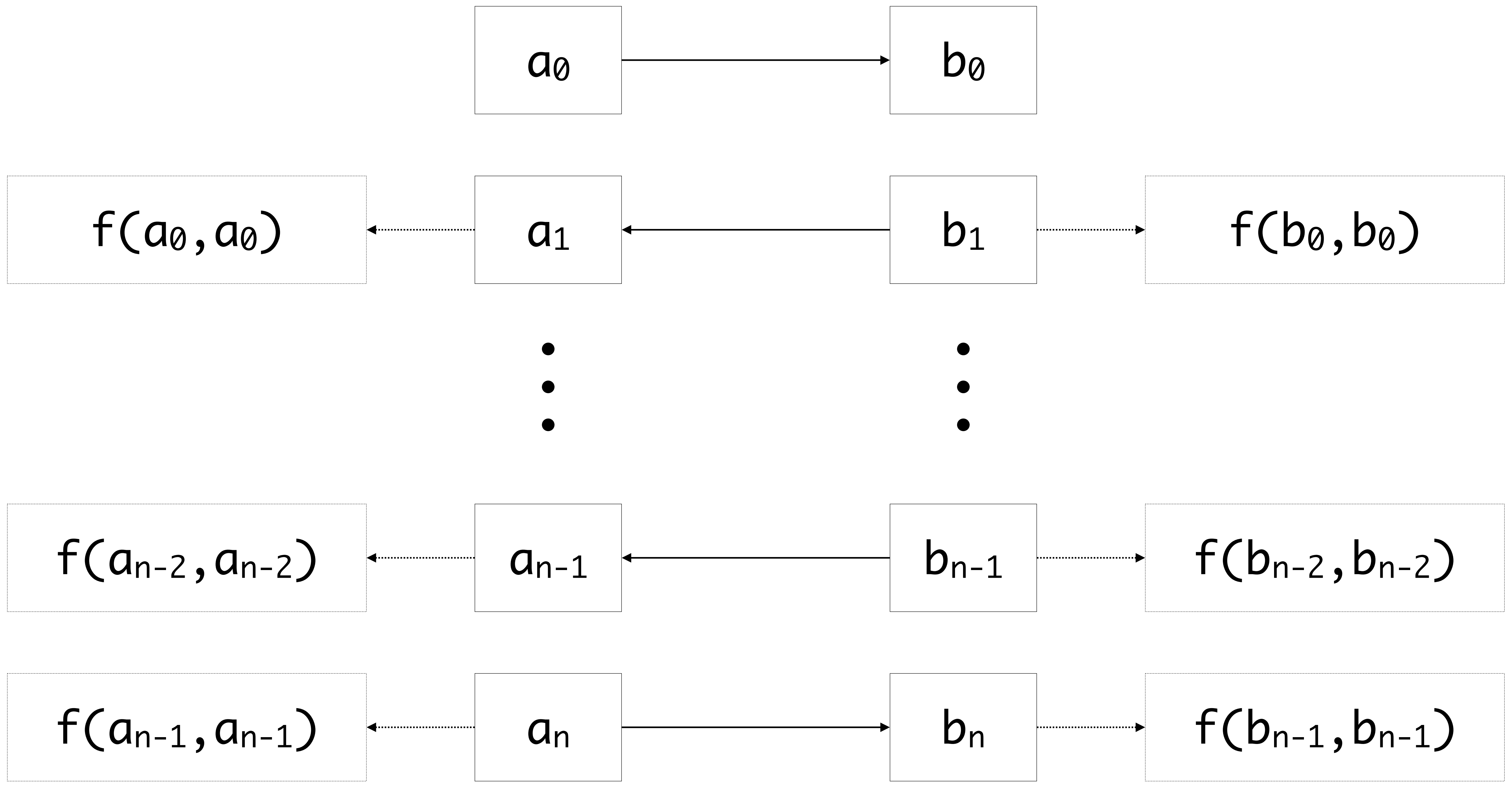

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How about occurrence checks?

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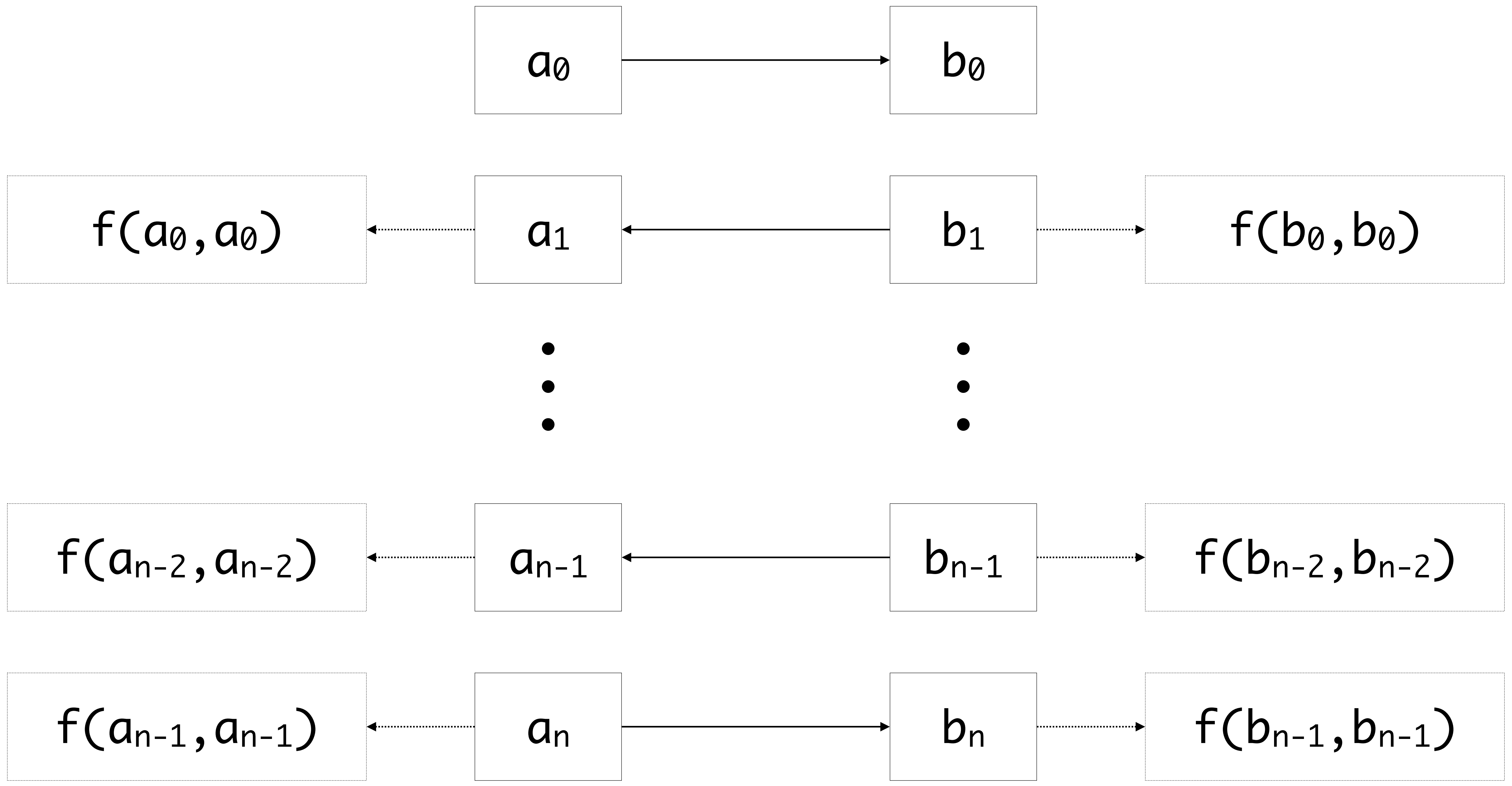

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How about occurrence checks? Postpone!

Union-Find

Main idea

Martelli, Montanari. An Efficient Unification Algorithm. TOPLAS, 1982

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- Postpone occurrence checks to prevent traversing (potentially) large terms

Conclusion

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What is the meaning of constraints?

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- Completeness: if a solution exists, the solver finds it
- Principality: the solver computes most general solutions